On claw-free t-perfect graphs

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Abstract

A graph is called t-perfect, if its stable set polytope is defined by non-negativity, edge and odd-cycle inequalities. We characterise the class of all claw-free t-perfect graphs by forbidden t-minors, and show that they are 3-colourable. Moreover, we determine the chromatic number of claw-free h-perfect graphs and give a polynomial-time algorithm to compute an optimal colouring.

1 Introduction

Perfect graphs can be determined by the structure of their stable set polytope. The *stable set polytope*, or SSP for short, is the convex hull of the characteristic vectors of independent vertex sets, the stable sets. In the case of a perfect graph, the SSP is fully described by non-negativity and clique inequalities. Vice versa, if the SSP of some graph is given by these types of inequalities then the graph is perfect.

In analogy to the relationship between perfect graphs and the SSP, Chvátal [8] proposed to investigate a class of graphs now called t-perfect: the class of graphs whose SSP is determined by non-negativity, edge and odd-cycle inequalities. (For precise definitions see next section.) The class of t-perfect graphs includes the series-parallel graphs (Boulala and Uhry [2]) and the almost bipartite graphs, i.e. those graphs that become bipartite upon deletion of a single vertex (Fonlupt and Uhry [14]). The latter result was extended by Shepherd [31], who determined which near-bipartite graphs are t-perfect. (A graph G is near-bipartite if $G-N(v)$ is bipartite for every vertex v.) Gerards and Shepherd [19] characterise the graphs with all subgraphs t-perfect. Also, t-perfect graphs share some computational properties with perfect graphs. For instance, the max-weight stable set problem can be solved efficiently for t-perfect graphs, see Grötschel, Lovász and Schrijver [21].

A prime example of a graph that is not t-perfect is the complete graph on four vertices, the K_4. Indeed, this graph will play an important role in what follows.

In this paper, we prove two theorems for t-perfect graphs that are, in addition, claw-free. We show that these graphs can be 3-coloured and we characterise them in terms of forbidden substructures.

Standard polyhedral methods assert that the fractional chromatic number of a t-perfect graph is at most 3. Shepherd suggested that t-perfect graphs might always be k-colourable for some fixed small k. As Laurent and Seymour found a t-perfect graph with $\chi = 4$ (see [30, p. 1207]), this number k has to be at least 4.

Conjecture 1. *Every t-perfect graph is 4-colourable.*

We prove that if the graphs are additionally claw-free then three colours suffice.

Theorem 2. *Every claw-free t-perfect graph is 3-colourable.*

Moreover, such a 3-colouring can be computed in polynomial time (Corollary 23).

We remark that compared to a result of Chudnovsky and Ovetsky [5] our Theorem 2 yields an improvement of 1. Indeed, Chudnovsky and Ovetsky show that the chromatic number of a quasi-line graph G is bounded by $\frac{3}{2}\omega(G)$. As no t-perfect graph can contain a clique of at least four vertices and, furthermore, as a claw-free t-perfect graph is quasi-line, Chudnovsky and Ovetsky's bound is applicable and yields $\chi \leq 4$ for all claw-free t-perfect graphs.

The celebrated strong perfect graph theorem of Chudnovsky, Robertson, Seymour and Thomas [7] characterises perfect graphs in terms of forbidden induced subgraphs: a graph is perfect if and only if it does not contain odd holes or anti-holes. We prove an analogous, although much more modest, result for claw-free t-perfect graphs. While, in order to describe perfect graphs, induced subgraphs are suitable as forbidden substructures, for t-perfect graphs a more general type of substructure, called a t-minor, is appropriate. Briefly, a t-minor is any graph obtained from the original graph by two kinds of operations, both of which preserve t-perfection: vertex deletions and simultaneous contraction of all the edges incident with a vertex whose neighbourhood forms an independent set. With this notion our second result is as follows.

Theorem 3. *A claw-free graph is t-perfect if and only if it does not contain any of K_4, W_5, C_7^2 and C_{10}^2 as a t-minor.*

Here, K_4 denotes the complete graph on four vertices, W_5 is the 5-wheel, and for $n \in \mathbb{N}$ we denote by C_n^2 the square of the cycle C_n on n vertices, see Figure 1. (The *square* of a graph is obtained by adding edges between any two vertices of distance 2.)

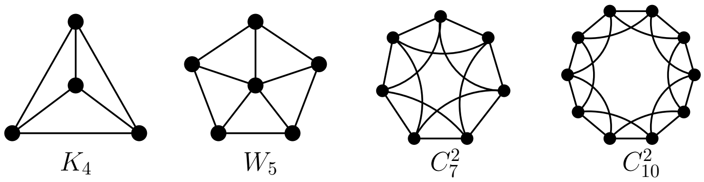

Figure 1: The forbidden t-minors.

The graphs from Theorem 3 already appear implicitly in Galluccio and Sassano [17]. They showed that every rank facet in a claw-free graph comes from a combination of inequalities describing cliques, line graphs of 2-connected factor-critical graphs, and circulant graphs $C_{\alpha\omega+1}^{\omega-1}$. However, as a claw-free graph may have non-rank facets we will not be able to make use of these results.

Ben Rebea's conjecture describes the structure of the stable set polytope of quasi-line graphs. As the conjecture has been solved (see Eisenbrand et al [13] and Chudnovsky and Seymour [6]), and as claw-free t-perfect graphs are quasi-line, it seems conceivable to use Ben Rebea's conjecture to prove Theorem 3. We have not pursued this approach for three reasons. First, Theorem 3 does not appear to be a direct consequence of the conjecture. Second, the solution of the conjecture rests on Chudnovsky and Seymour's characterisation of claw-free graphs, which is far from trivial. Finally, our proof of Theorem 3 (with a little extra effort) yields a 3-colouring of claw-free t-perfect graphs.

Let us give a brief outline of the paper. In the next section, we give a more formal definition of t-perfect graphs and introduce some of their properties. In Section 3 we determine which squares of cycles are t-perfect, and prove our main tool for the proofs of both our main results. This tool, Lemma 9, says that every claw-free t-perfect 3-connected graph is a line graph or one of three exceptional graphs. We prove Theorem 3 in Section 4, and exhibit some open problems from the same direction. In Section 5 we will prove Theorem 2 and develop an algorithm to compute a 3-colouring for any claw-free t-perfect graph. These two results will be extended to the larger class of claw-free h-perfect graphs in the last section.

2 Definition of t-perfect graphs

All our graphs are finite and simple, so we do not allow parallel edges or loops. For general graph-theoretic concepts and notation we refer the reader to Diestel [10], for more on t-perfect and claw-free graphs to Schrijver [30, Chapters 68 and 69].

Let $G = (V, E)$ be a graph. The *stable set polytope* $\mathrm{SSP}(G) \subseteq \mathbb{R}^V$ of G is defined as the convex hull of the characteristic vectors of stable, i.e. independent, subsets of V. We define a second polytope $\mathrm{TSTAB}(G) \subseteq \mathbb{R}^V$ for G, given by

$$\begin{aligned} &x \geq 0, \\ &x_u + x_v \leq 1 \text{ for every edge } uv \in E, \\ &x(C) \leq \lfloor |C|/2 \rfloor \text{ for every induced odd cycle } C \text{ in } G. \end{aligned} \tag{1}$$

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, it holds that $\mathrm{SSP}(G) \subseteq \mathrm{TSTAB}(G)$.

We say that the graph G is *t-perfect* if $\mathrm{SSP}(G)$ and $\mathrm{TSTAB}(G)$ coincide. Equivalently, G is t-perfect if and only if $\mathrm{TSTAB}(G)$ is an integral polytope, i.e. if all its vertices are integral vectors.

Neither the complete graph on four vertices K_4 nor the 5-wheel W_5 are t-perfect. Indeed, for K_4 the vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ lies in TSTAB but not in the SSP of K_4 as the sum over all entries is larger than $\alpha(K_4) = 1$. The vector that assigns a value of $\frac{2}{5}$ to each vertex on the rim and a value of $\frac{1}{5}$ to the centre shows that 5-wheel is t-imperfect. Again, the vector lies in TSTAB but the sum of all entries is larger than $\alpha(W_5) = 2$.

The following fact is well-known:

$$\textit{bipartite graphs are } t\textit{-perfect.} \tag{2}$$

In fact, the SSP of a bipartite graph is fully described by just non-negativity and edge inequalities.

It is easy to check that vertex deletion preserves t-perfection (edge deletion, however, does not). A second operation that maintains t-perfection is described in Gerards and Shepherd [19]:

for a vertex v for which $N(v)$ is a stable set contract all edges in $E(v)$. $(*)$

We call this operation a *t-contraction at v*. Let us say that H is a *t-minor* of G if it is obtained from G by repeated vertex-deletion and t-contraction. Then, if G is t-perfect, so is H. We call a graph *minimally t-imperfect* if it is not t-perfect but every proper t-minor of it is t-perfect. Obviously, in order to characterise t-perfect graphs in terms of forbidden t-minors it suffices to find all minimally t-imperfect graphs.

The following simple lemma ensures that we stay within the class of claw-free graphs when taking t-minors. (For a proof, observe that a claw in a t-minor can only arise from an induced subdivided claw in the original graph.)

Lemma 4. *Every t-minor of a claw-free graph is claw-free.*

3 Reduction to line graphs

In order to prove the back direction of Theorem 3, let us assume we are given a claw-free t-imperfect graph G. Our task is then to find K_4, W_5, C_7^2 or C_{10}^2 as a t-minor of G. In this section we show that the problem reduces to line graphs if G is, in addition, 3-connected.

One case that we need to deal with is when G is the square of a cycle. Recall that we denote by C_n^2 the square of a cycle of order n, that is C_n^2 is the graph on the vertex set $\{v_1, \ldots, v_n\}$ so that $v_iv_j \in E(C_n^2)$ if and only if $|i-j| \bmod n \in \{1, 2\}$.

The only two squares of cycles that are t-perfect are the triangle $C_3^2 = C_3$ and C_6^2. This can be shown either directly or by making use of the results of Dahl [9], who gave a complete description of the stable set polytope of C_n^2. Knowing which C_n^2 are t-perfect and which are not, however, is not enough for our purpose. We will need to know that every t-imperfect C_n^2 contains one of the four forbidden t-minors. This is the aim of the first two lemmas.

Lemma 5. *Let $n \geq 5$, and let $n \notin \{6, 7, 10\}$. Then K_4 is a t-minor of $C_n^2 - v_5$.*

Proof. Depending on n mod 4 we perform vertex-deletions and then t-contractions as indicated in Figure 2 until the only vertices left are $v_1, \ldots, v_4$. In particular, we delete the grey vertices in the initial segment (marked by a dashed box). Outside this segment we delete every other vertex until we reach the first vertex v_1 again. Finally, we contract the odd path between v_4 and v_1 to a single edge.

The length of the initial segment poses a constraint on the minimal size of the graph. For $n \equiv 0 \pmod 4$ the construction is possible for $n \geq 8$, for $n \equiv 1$ we need $n \geq 5$, for $n \equiv 2$ we need $n \geq 14$, and $n \geq 11$ is necessary for $n \equiv 3$. So the only cases we have not dealt with are $n = 6, 7, 10$, which are precisely the exceptions. □

Lemma 6. *C_7^2 and C_{10}^2 are minimally t-imperfect.*

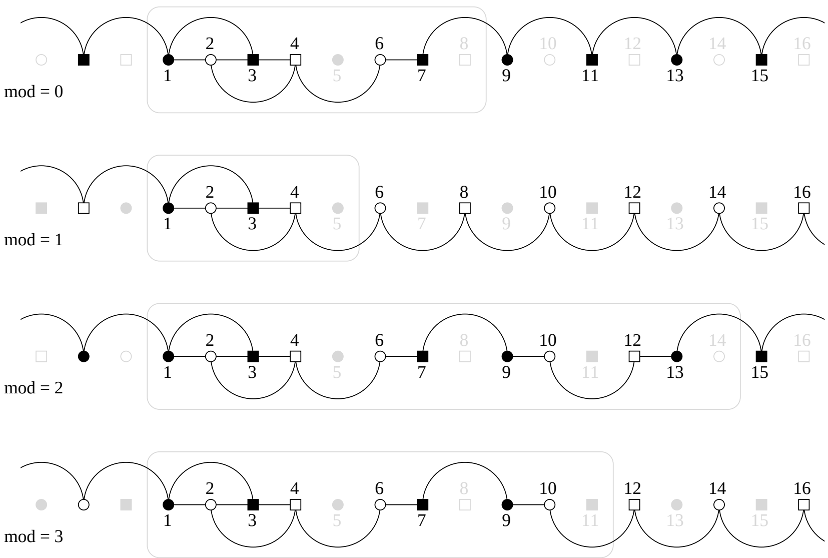

Figure 2: K_4-t-minors in C_n^2 depending on n mod 4.

Proof. For $i \in \{7, 10\}$ define the vector $x \in \mathbb{R}^{V(C_i^2)}$ by setting $x_v = 1/3$ for each $v \in V(C_i^2)$. Then x clearly lies in $\text{TSTAB}(C_i^2)$. However, $x \notin \text{SSP}(C_i^2)$ as $\mathbf{1}^T x = i/3 > \lfloor \frac{i}{3} \rfloor = \alpha(C_i^2)$. Thus, neither of C_7^2 and C_{10}^2 is t-perfect.

To see that C_7^2 and C_{10}^2 are minimally t-imperfect we only need to check that for $i = 7, 10$ the graph $C_i^2 - v_1$ is t-perfect. From [9, Theorem 4.3] it follows that the stable set polytope of C_i^2 is described by[1] the nonegativity constraints, the triangle inequalities and the inequality $x(V(C_i^2)) \leq i/3$.

The stable set polytope of $C_i^2 - v_1$ coincides with $\text{SSP}(C_i^2)$ restricted to $x_{v_1} = 0$. But when $x_{v_1} = 0$, the inequality $x(V(C_i^2)) \leq i/3$ is redundant as it is implied by two resp. three triangle inequalites, namely the inequalities for the triangles $v_2v_3v_4$, $v_5v_6v_7$ and, if $i = 10$, also $v_8v_9v_{10}$. Hence $\text{TSTAB}(C_i^2 - v_1) \subseteq \text{SSP}(C_i^2 - v_1)$, and thus the two polytopes coincide, as desired. $\square$

We need a result by Harary. Let us call a triangle T *odd* if there is a vertex v outside T that is adjacent to an odd number of the vertices in T.

Theorem 7 (Harary [23, Theorem 8.4])**.** *Let G be a claw-free graph. Then G is a line graph (of a simple graph) if and only if every pair of odd triangles that shares exactly one edge induces a K_4.*

We now prove the main lemma in our reduction to line graphs.

Lemma 8. *Let G be a 3-connected claw-free graph with $\Delta(G) \leq 4$. If G does not contain K_4 as t-minor then one of the following statements holds true:*

(i) *G is a line graph;*

(ii) *$G \in \{C_6^2 - v_1v_6, C_7^2 - v_7, C_{10}^2 - v_{10}, C_7^2, C_{10}^2\}$.*

[1]Note that in our two cases Dahl's 1-interval inequalities do not occur.

Proof. We shall repeatedly make use of the following argument. Assume that in the neighbourhood of a vertex u we find a path xyz, and assume that u has a fourth neighbour $v \notin \{x, y, z\}$. As K_4 is not a subgraph of G we know that $xz \notin E(G)$. Then, because G is claw-free, v has to be adjacent to x or to z or to both.

First of all, we shall show that

$$P_6^2 \textit{ is a subgraph of } G. \tag{3}$$

Recall that P_k denotes a path on k vertices, and that P_k^2 denotes the square of P_k, i.e. the graph obtained from P_k by adding an edge between any two vertices of distance 2.

Indeed, as we may assume that G is not a line graph, there exist by Theorem 7 two odd triangles that share exactly one edge, say $u_1u_2u_3$ and $u_2u_3u_4$. As G is 3-connected, $\{u_1, u_4\}$ does not separate G, and thus one of u_2 and u_3 has a neighbour $u_5 \notin \{u_1, u_2, u_3, u_4\}$. By symmetry, we may assume that $u_3u_5 \in E(G)$ and by the argument outlined at the beginning of this proof, we deduce from $u_1u_2u_4 \subseteq G[N(u_3)]$ that u_5 is adjacent to u_1 or to u_4 (or to both). Symmetry, again, allows us to assume that u_5 is adjacent to u_4.

As K_4 is not a subgraph of G, u_1 and u_5 each send exactly two edges to the triangle $u_2u_3u_4$. That triangle, however, is odd. Thus there exists a vertex $u_6 \notin \{u_1, \ldots, u_5\}$ that is adjacent to an odd number of vertices of the triangle. Since u_3 has four neighbours already among the u_i, it follows that u_6 is either adjacent to u_2 or to u_4. First, assume that $u_6u_4 \in E(G)$, and that $u_6u_2 \notin E(G)$. The path $u_2u_3u_5$ that is contained in the neighbourhood of u_4 together with $u_6u_2 \notin E(G)$ ensures that u_6 is adjacent to u_5. On the other hand, if $u_6u_4 \notin E(G)$, but $u_6u_2 \in E(G)$, then we may argue in the same way to obtain that u_6 is adjacent to u_1. In either case, this proves (3).

Next, we prove that

$$\textit{if } k \geq 6 \textit{ so that } P_k^2 \subseteq G, \textit{ then either } P_{k+1}^2 \subseteq G \textit{ as well, or } V(G) = V(P_k). \tag{4}$$

Assume that G has a vertex outside $P_k = v_1 \ldots v_k$. Because G is 3-connected and $\Delta(G) \leq 4$, one of v_2 and v_{k-1}, let us say the latter, has a neighbour $v_{k+1} \notin V(P_k)$; if not then v_1 and v_k would separate $V(P_k)$ from the rest of the graph. From the fact that the path $v_{k-3}v_{k-2}v_k$ is contained in the neighbourhood of v_{k-1} we deduce that v_{k+1} is adjacent to v_{k-3} or to v_k. However, v_{k-3} is already adjacent to four vertices, namely to $v_{k-5}, v_{k-4}, v_{k-2}, v_{k-1}$ (recall that $k \geq 6$). Thus, $\Delta(G) \leq 4$ implies that v_{k+1} is in fact adjacent to v_k. Thus $P_{k+1}^2 \subseteq G$ and we have proved (4).

Now, by repeated application of (4) we arrive at a path $P_n = v_1 \ldots v_n$, for some $n = |V(G)| \geq 6$, whose square is a subgraph of G. Observe that in the square of P_n every vertex has degree 4, except v_2 and v_{n-1}, which have degree 3, and except v_1 and v_n, which have degree 2. Since $\Delta(G) \leq 4$, the square of P_n and G may only differ in the presence or absence of the edges v_1v_{n-1}, v_1v_n, v_2v_{n-1} and v_2v_n in G. As G is 3-connected, each of v_1 and v_n is incident with at least one of these edges.

First, assume that $v_1v_n \notin E(G)$, which immediately entails that $v_1v_{n-1} \in E(G)$ and $v_2v_n \in E(G)$, and hence, as $\Delta(G) \leq 4$, that $v_2v_{n-1} \notin E(G)$. Since

$v_1v_3v_4$ is a path in the neighbourhood of v_2, the fourth neighbour v_n of v_2 must be adjacent to v_4. This is only possible if $n = 6$, and we find that then $G = C_6^2 - v_1v_6$, which is as desired.

So, from now on, let us assume that

$$v_1v_n \in E(G). \tag{5}$$

Next, suppose that v_2v_{n-1} is an edge of G. Then $n > 6$ as otherwise $v_2, v_3, v_4, v_5 = v_{n-1}$ span a K_4. On the other hand, we find the path $v_{n-3}v_{n-2}v_n$ in the neighbourhood of v_{n-1}, which implies that v_2 is adjacent to v_{n-3} or to v_n. Since v_2 already has four neighbours, namely v_1, v_3, v_4 and v_{n-1}, and since $n > 6$ it follows that $v_{n-3} = v_4$ and $n = 7$.

Consequently, G is isomorphic to $\tilde{C}_7^2$, which we define as the square of P_7 plus the edges v_1v_7 and v_2v_6. However, Figure 3 A shows that $\tilde{C}_7^2$ contains K_4 as a t-minor, a contradiction. (Alternatively, we might have argued that $\tilde{C}_7^2$ is the line graph of the graph obtained from K_4 by subdividing one edge.)

Thus,

$$v_2v_{n-1} \notin E(G). \tag{6}$$

So, by (5) and (6), G is isomorphic to one of the following graphs: $G = C_n^2$, $C_n^2 - v_1v_{n-1}$, and $C_n^2 - v_1v_{n-1} - v_2v_n$. Let us check these cases seperately.

First, assume $G = C_n^2$. Since $C_6^2 = L(K_4)$ and since by Lemma 5, for $n \geq 7$ every C_n^2 except C_7^2 and C_{10}^2 contains K_4 as a t-minor, we find that $G = C_7^2$ or $G = C_{10}^2$, which are two of the allowed outcomes of Lemma 8.

Next, assume that $G = C_n^2 - v_1v_{n-1}$. Observe that $(C_n^2 - v_1v_{n-1}) - v_1$ is isomorphic to $C_n^2 - v_5$. Hence, unless $n \in \{6, 7, 10\}$, Lemma 5 asserts that G contains K_4 as a t-minor. For $n = 7$ and $n = 10$, Figure 3 B and C indicate K_4-t-minors of G. So, $n = 6$, that is, $G = C_6^2 - v_1v_5$ which is isomorphic to $C_6^2 - v_1v_6$, and thus one of the allowed outcomes of the lemma.

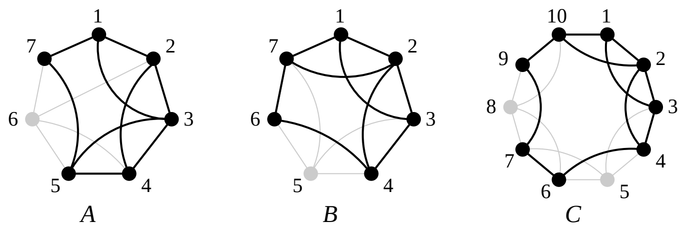

Figure 3: K_4 as a t-minor of $\tilde{C}_7^2$, $C_7^2 - v_1v_6$, $C_{10}^2 - v_1v_9$.

Finally, we treat the case when $G = C_n^2 - v_1v_{n-1} - v_2v_n$. Observe that then G is isomorphic to $C_{n+1}^2 - v_{n+1}$, and thus we may employ Lemma 5 again to deduce that $n + 1 \in \{6, 7, 10\}$. Of these cases, $n + 1 = 6$ is impossible as $n \geq 6$ by (3). Therefore, either $G = C_7^2 - v_7$ or $G = C_{10}^2 - v_{10}$, which is as desired. □

We note the following consequence.

Lemma 9. *Let G be a 3-connected claw-free graph. If G does not contain any of K_4, W_5, C_7^2 or C_{10}^2 as a t-minor then G is a line graph or $G \in \{C_6^2 - v_1v_6, C_7^2 - v_1, C_{10}^2 - v_1\}$.*

Proof. Suppose G has a vertex v of degree 5 or higher. If v has at least six neighbours, then the subgraph of G induced by $N(v)$ contains a triangle or three independent vertices. Both cases are impossible, as the former leads to a K_4, and the latter to a claw. On the other hand, if $|N(v)| = 5$ then $G[v \cup N(v)]$ is a 5-wheel, unless it contains a claw or a K_4. Thus, we have $\Delta(G) \leq 4$ and the lemma follows from Lemma 8. □

4 Characterising claw-free t-perfect graphs

In the previous section we have seen that we may reduce our problem to line graphs if we can additionally assume 3-connectivity. The task of the next few lemmas is to show that minimally t-imperfect claw-free graphs are, in fact, 3-connected.

The first of these lemmas is quite similar to Lemma 12 in Gerards and Shepherd [19]. As that lemma, however, is assembled from results of various authors, its proof is not easily verified. We therefore give a direct proof that draws on only two fairly simple facts.

Lemma 10. *Let G be a minimally t-imperfect graph, and assume $u, v \in V(G)$ to separate G. Then $G - \{u, v\}$ has exactly two components, one of which together with u and v induces a path in G. Moreover, $uv \notin E(G)$.*

Proof. Let $G = G_1 \cup G_2$ so that $\{u, v\} = V(G_1) \cap V(G_2)$ and such that $\{u, v\}$ separates $G_1 - \{u, v\}$ from $G_2 - \{u, v\}$. Suppose that neither of G_1 and G_2 is a path. Let z be a non-integral vertex of TSTAB(G), denote by $\mathcal{I}$ the set of those non-negativity, edge and odd-cycle inequalities that are satisfied with equality by z. We define z^1 resp. z^2 to be the restriction of z to G_1 resp. G_2.

As in the proof of Theorem 1 in Gerards and Shepherd [19] we can deduce that

$$0 < z_w < 1 \textit{ for all } w \in V(G) \tag{7}$$

and

$$\textit{every odd cycle whose inequality is in } \mathcal{I} \textit{ fails to separate } G. \tag{8}$$

The last fact implies, in particular, that each odd cycle in $\mathcal{I}$ lies either completely in G_1 or in G_2 (recall that neither of G_1 and G_2 is a path). Thus, we can partition $\mathcal{I}$ in $(\mathcal{I}_1, \mathcal{I}_2)$ so that $\mathcal{I}_k$ pertains only to G_k. Now, if there is a $j \in \{1, 2\}$ so that $\dim \mathcal{I}_j = |V(G_j)|$ then z^j is a vertex of TSTAB(G_j) =SSP(G_j). Since z^j is non-integral we obtain a contradiction.

Therefore, we have $\dim \mathcal{I}_k = |V(G_k)| - 1$ for $k = 1, 2$, which means that $\mathcal{I}_k$ describes an edge of TSTAB(G_k). Denote the endvertices of this edge by s^k and t^k, i.e. $z^k = \lambda_k s^k + (1 - \lambda_k) t^k$ for some $0 \leq \lambda_k \leq 1$. As TSTAB(G_k) =SSP(G_k) by assumption, it follows that s^k is the characteristic vector of a stable set S_k of G_k; the same holds for t^k and a stable set T_k.

By (7), $z^1_u = z^2_u > 0$ and thus for each $k = 1, 2$ one of S_k and T_k needs to contain u. By renaming if necessary we may assume that $u \in S_1$ and $u \in S_2$. Then $u \notin T_k$ for $k = 1, 2$ as otherwise we obtain $z^k_u = \lambda_k + (1 - \lambda_k) = 1$ in contradiction to (7). This implies that

$$\lambda_1 = z^1_u = z_u = z^2_u = \lambda_2. \tag{9}$$

If $S_1 \cap \{v\} = S_2 \cap \{v\}$ then also $T_1 \cap \{v\} = T_2 \cap \{v\}$ as (7) implies as above that $v \in S_k$ if and only if $v \notin T_k$. In this case, $S := S_1 \cup S_2$ and $T := T_1 \cup T_2$ are stable sets of G and we obtain $z = \lambda_1 \chi_S + (1 - \lambda_1)\chi_T$, contradicting the choice of z as a non-integral vertex of TSTAB(G).

So, let us assume that S_1 and S_2 differ on $\{v\}$. Without loss of generality, let $v \in S_1$ but $v \notin S_2$. Then

$$\begin{aligned} S_1 \cap \{u, v\} &= \{u, v\}, & T_1 \cap \{u, v\} &= \emptyset, \\ S_2 \cap \{u, v\} &= \{u\} \quad \textit{and} & T_2 \cap \{u, v\} &= \{v\}. \end{aligned}$$

So, $\lambda_1 = z_v^1 = z_v^2 = 1 - \lambda_2$, and hence, by (9), $\lambda_1 = \lambda_2 = 1/2$. In particular, it follows with (7) again that $z_w = 1/2$ for all $w \in V(G)$.

Now, since bipartite graphs are t-perfect by (2), G contains an odd cycle of length $2k + 1$, say. However, adding up z along the cycle yields $k + 1/2$, contradicting the odd-cycle inequalities. □

Next, let us prove that a minimally t-imperfect claw-free graph has minimum degree at least three. We start with a lemma that is a variant of Theorem 2.5 in Barahona and Mahjoub [1], and can be proved in a very similar way.

Lemma 11 (Barahona and Mahjoub [1])**.** *Let G be a graph, and let uvw be a path in G so that $\deg(v) = 2$ and $uw \notin E(G)$. Furthermore, let $a^T x \leq \alpha$ be a facet-defining inequality of SSP(G) so that $a_u = a_v = a_w$. Denote by G' the graph obtained from G by contracting uv and vw, and let $\tilde{v}$ be the resulting vertex, i.e. $V(G') \setminus V(G) = \{\tilde{v}\}$. If $a' \in \mathbb{R}^{V(G')}$ is defined by $a'_p = a_p$ for $p \in V(G' - \tilde{v})$ and $a'_{\tilde{v}} = a_v$ then $a'^T x \leq \alpha - a_v$ is a facet-defining inequality of SSP(G').*

The following lemma serves to guarantee that $a_u = a_v = a_w$ as in Lemma 11.

Lemma 12. *Let G be a graph and assume that for $a \in \mathbb{R}^{V(G)}$, $a > 0$ the inequality $a^T x \leq \alpha$ is facet-defining in SSP(G), and that it is not a multiple of an edge inequality or of an odd-cycle inequality.*

(i) *If G contains a path uvw so that $\deg(v) = 2$ then $a_v \leq a_w$.*

(ii) *If G contains a triangle wpq and a neighbour $v \notin \{p, q\}$ of w so that $\deg(w) = 3$ then $a_v \geq a_w$.*

Assertion (i) appears in Mahjoub [25].

Proof. For both cases, observe that as the SSP is full-dimensional there exists a set $\mathcal{S}$ of $|V(G)|$ affinely independent stable sets that satisfy $a^T x \leq \alpha$ with equality. Since $a > 0$ it follows that $\alpha \neq 0$, which, in turn, implies that the characteristic vectors of the stable sets in $\mathcal{S}$ are even linearly independent. In particular, any inequality satisfied with equality by all $S \in \mathcal{S}$ is a multiple of $a^T x \leq \alpha$.

(i) Since $a^T x \leq \alpha$ is not a multiple of the edge inequality $x_u + x_v \leq 1$ there must exist an $S_0 \in \mathcal{S}$ so that $u \notin S_0$ and $v \notin S_0$. As $a > 0$ this implies that $w \in S_0$. Clearly, $S'_0 := S_0 \setminus \{w\} \cup \{v\}$ is a stable set and thus $a^T \chi_{S'_0} \leq \alpha = a^T \chi_{S_0}$. Hence $a_v \leq a_w$.

(ii) Since $a^T x \leq \alpha$ is not a multiple of the triangle inequality $x_w + x_p + x_q \leq 1$ there must exist an $S_1 \in \mathcal{S}$ so that $\{w, p, q\} \cap S_1 = \emptyset$. Then, as $a > 0$ and

$N(w) = \{v, p, q\}$, we have that $v \in S_1$ and that $S_1' := S_1 \setminus \{v\} \cup \{w\}$ is stable. Again, we obtain $a^T \chi_{S_1'} \leq \alpha = a^T \chi_{S_1}$ and therefore $a_w \leq a_v$. □

Lemma 13. *Let G be a minimally t-imperfect claw-free graph. Then G has minimum degree ≥ 3.*

Proof. It is easy to see that no vertex can have degree 1. Indeed, such a vertex would lead to a violation as in (7). Also, it cannot happen that all vertices have degree 2, as then G would be a cycle, and thus t-perfect. So suppose there is a path $P = w_1 \dots w_k$ with $k \geq 3$ so that all internal vertices have degree 2 in G but w_1 and w_k have degree > 2. Since G is claw-free and does not properly contain a K_4 we deduce that $\deg(w_1) = \deg(w_k) = 3$, and in fact there are neighbours p_1, q_1 of w_1 and p_k, q_k of w_k so that $w_1p_1q_1$ and $w_kp_kq_k$ are triangles in G.

As G is t-imperfect there exists a facet-defining inequality $a^T x \leq \alpha$ of $\mathrm{SSP}(G)$ with $a \geq 0$ that is not a multiple of a non-negativity, edge or odd-cycle inequality. Since G is minimally t-imperfect under vertex deletion it follows furthermore that $a > 0$.

Now, applying (i) of Lemma 12 we get that $a_{w_2} = \ldots = a_{w_{k-1}} \leq \min\{a_{w_1}, a_{w_k}\}$. Then, (ii) yields that $a_{w_1} = a_{w_2} = \ldots = a_{w_k}$.

Denote by G' the graph obtained from G by performing a t-contraction at w_2, and let $\tilde{w}$ be the resulting new vertex. Define $a'_u = a_u$ for $u \in V(G' - \tilde{w})$ and $a'_{\tilde{w}} = a_{w_2}$. Then, by Lemma 11, $a'^T x \leq \alpha - a_{w_2}$ is facet-defining for $\mathrm{SSP}(G')$. However, as $a' > 0$ and as G' is t-perfect it follows that G' consists of a single vertex, a single edge or of a single odd cycle. Then G is such a graph, too, and thus t-perfect, a contradiction. □

The final step towards the proof of Theorem 3 consists in describing t-perfect line graphs in terms of forbidden t-minors. Cao and Nemhauser [3], among other results, already characterise t-perfect line graphs in terms of forbidden subgraphs. Unfortunately, their characterisation appears erroneous. While we therefore cannot make use of their theorem, we will pursue an approach that is inspired by their work. In particular, we take advantage of Edmonds [11] celebrated theorem on the matching polytope.

For a graph G, define the *matching polytope* $M(G) \subseteq \mathbb{R}^{E(G)}$ to be the convex hull of the characteristic vectors of matchings. Recall that a graph G is *factor-critical* if $G - v$ has a perfect matching for every vertex v.

Theorem 14 (Edmonds [11], Pulleyblank and Edmonds [27])**.** *Let G be a graph and $x \in \mathbb{R}^{E(G)}$. Then $x \in M(G)$ if and only if*

$$x \geq 0 \qquad (10)$$

$$\sum_{e \in E(v)} x_e \leq 1 \qquad \textit{for each } v \in V(G) \qquad (11)$$

$$\sum_{e \in E(F)} x_e \leq \left\lfloor \frac{|V(F)|}{2} \right\rfloor \qquad \textit{for each 2-connected factor-critical } F \subseteq G. \qquad (12)$$

We say that G has a *proper odd ear decomposition* if there is a sequence $G_0, G_1, \ldots, G_n$ so that G_0 is an odd cycle, $G_n = G$ and G_k is obtained from G_{k-1} for $k = 1, \ldots, n$ by adding an odd path between two (distinct) vertices of G_{k-1} whose interior vertices are disjoint from G_{k-1}.

Theorem 15 (Lovász [24]). *A graph is 2-connected and factor-critical if and only if it has a proper odd ear-decomposition.*

For the proof of the next two lemmas, we need the following auxiliary definition. Consider a Θ-graph G, i.e. a graph consisting of three non-crossing paths with common endvertices. If exactly two of these paths have odd length, we shall call G an *ooe-Θ-graph.* Note that G_1 in the odd ear decomposition above constitutes such an ooe-Θ-graph.

Lemma 16. *The following statements are equivalent for a graph G:*

(i) *The line graph $L(G)$ is t-perfect.*

(ii) *$L(G)$ does not contain K_4 as a t-minor.*

(iii) *$\Delta(G) \leq 3$ and no subgraph of G is an ooe-Θ-graph.*

Proof. (i)→(ii) is clear. For (ii)→(iii) note that a vertex of degree at least 4 in G leads to a K_4-subgraph of $L(G)$. Furthermore, an ooe-Θ-graph in G gives rise to an induced subgraph of $L(G)$ that yields a K_4-t-minor.

In order to show (iii)→(i) let us assume that $H := L(G)$ satisfies (iii). Since $M(G) = \mathrm{SSP}(H)$, all we have to show is that $\mathrm{TSTAB}(H)$ is a subset of the polytope described by (10), (11), and (12) from Theorem 14. That is, we have to check that the inequalities from Theorem 14 are valid for $\mathrm{TSTAB}(H)$.

Condition (10) is clear, and for (11), pick a (non-isolated) vertex v of G. If v has degree 2 then (11) follows from an edge inequality in H, and if $d(v) = 3$ then (11) follows from an odd-cycle inequality for a triangle.

For (12), suppose that G contains a 2-connected factor-critical subgraph F, which, by Theorem 15, has an odd ear-decomposition. If F is not an odd cycle then F contains an ooe-Θ-graph, which we have excluded. Hence F is an odd cycle, and (12) follows from some odd-cycle inequality in H. Thus, we have shown that $\mathrm{SSP}(H)$ coincides with $\mathrm{TSTAB}(H)$, as desired. □

We now restate and prove our main result.

Theorem 3. *A claw-free graph is t-perfect if and only if it does not contain any of K_4, W_5, C_7^2 and C_{10}^2 as a t-minor.*

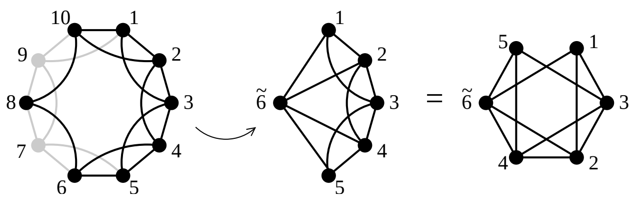

Figure 4: $C_6^2 - v_1v_6$ is a t-minor of $C_{10}^2 - v_9$.

Proof. As neither of K_4, W_5, C_7^2 and C_{10}^2 is t-perfect (note Lemma 6), necessity is obvious. To prove sufficiency, consider a claw-free and minimally t-imperfect graph G. Lemmas 10 and 13 ensure that G is 3-connected, and by Lemma 9 G is either a line graph or $G \in \{C_6^2 - v_1v_6, C_7^2 - v_1, C_{10}^2 - v_1\}$. If G is a line graph then the theorem follows from Lemma 16. Since $C_7^2 - v_1$ and $C_{10}^2 - v_1$

are t-perfect as C_7^2 and C_{10}^2 are minimally t-imperfect (Lemma 6), it remains to consider the case when $G = C_6^2 - v_1v_6$. However, this graph is a t-minor of the t-perfect graph $C_{10}^2 - v_9 \cong C_{10}^2 - v_1$ as can be seen in Figure 4. □

Let us conclude this section with three open problems. Since odd holes and anti-holes in a graph can be detected in polynomial time, see Chudnovsky et al [4], the strong perfect graph theorem implies that perfect graphs can be recognised in polynomial time. A similar result for t-perfect graphs would be desirable:

Question 17. *Is there an algorithm with polynomial running time to test whether a given (claw-free) graph is t-perfect?*

A closer inspection of our proofs reveals that for such an algorithm it would be enough to detect K_4-t-minors in polynomial time. We note that Gerards [18] describes an algorithm with polynomial running time that tests whether a graph contains an odd-K_4 subdivision as a subgraph. (An odd-K_4 subdivision is a subdivision of K_4 in which each of the four triangles has become an odd cycle.)

In Theorem 3 we have determined all minimally t-imperfect graphs that are claw-free. To find all minimally t-imperfect graphs, with or without claws, seems a daunting task. Indeed, the class of these graphs already includes two infinite families, namely the odd wheels and the even Möbius ladders, see Shepherd [31]. Nevertheless, the examples that we know suggest two properties that all minimally t-imperfect graphs might share:

Question 18. *Are all minimally t-imperfect graphs* 3*-connected?*

In the light of Lemma 10, for an affirmative answer to this question, it would suffice to prove that all minimally t-imperfect graphs have minimum degree 3.

Question 19. *For a minimally t-imperfect graph G, does TSTAB(G) have precisely one non-integral vertex?*

We note that this is false if G is only minimal subject to vertex deletion. An example for this is the K_4 with one edge replaced by a path of length 3.

5 Colouring t-perfect graphs

The last two sections of the paper are devoted to colouring problems. We start with colouring claw-free t-perfect graphs in this section, and move to h-perfect graphs in the following section.

Let G be a graph with an edge-colouring, and let v be a vertex of G so that none of its incident edges are coloured with β. Then the *α/β-path from v* is the unique maximal path starting in v whose edges alternate between colours α and β.

Lemma 20. *Let G be a graph with $\Delta(G) \leq 3$. If G does not contain any ooe-Θ-graph then its chromatic index is at most* 3.

Proof. Suppose the statement to be false, and let G be a counterexample with smallest number of edges. Pick any edge $e = uv$, and suppose $E(G - e)$ is coloured with $\{1, 2, 3\}$. The edge e needs to be adjacent to edges of every colour as otherwise we could colour e with the free colour. So, since $\Delta(G) \leq 3$ we may assume that u is incident with precisely two edges f_1 and f_2 of colours 1 and 2, respectively, and that v is incident with an edge g_3 of colour 3 but not with any edge of colour 2. (There might be a third edge adjacent to v, which then has colour 1.)

Next, denote by P the 2/3-path from u. If P does not contain v we can swap the colours along P to obtain an edge-colouring of $G-e$ such that no edge incident with u or v is coloured with 2. Thus, assigning colour 2 to e yields an edge-colouring of G with 3 colours, in contradiction to our assumption. Hence, P contains v, and in fact ends there. Let us point out that $C := P + e$ is an odd cycle (possibly a triangle), as the first and the last edge of P have different colours.

Consider the 1/3-path Q_1 from u, and suppose that P and Q_1 have a vertex other than u in common. Denote by q_1 the first vertex on Q_1 after u that lies on P. Since the edges of P are coloured with 2 and 3, the edge preceeding q_1 on Q_1 needs to have colour 1, which implies that uQ_1q_1 is an odd path. Now, however, we have found three non-crossing paths linking u and q_1: namely the two paths joining u and q_1 contained in C, and uQ_1q_1. As C is odd, exactly two of these paths have odd length, and hence their union forms an ooe-Θ-graph, which is impossible.

So, we may assume that $Q_1 - u$ is disjoint from P. This means that we can swap colours along Q_1 without changing the colours on P. We thus arrive at an edge-colouring of $G - e$ where f_1 is coloured with 3, while all other edges incident with u or v keep their old colours. Since we have assumed that v is not incident with any edge of colour 2, we deduce that there exists an edge g_1 with endvertex v and colour 1; otherwise we could use 1 to colour e. Now we proceed as in the previous paragraph. If the 1/2-path Q_2 from v meets P only in v then by swapping colours along Q_2 we obtain a colouring of $E(G - e)$ in which no edge adjacent to e has colour 1, a contradiction. On the other hand, if there is a vertex $q_2 \neq v$ in $Q_2 \cap P$, which we may choose so that no interior vertex of vQ_2q_2 lies in P, then $C \cup vQ_2q_2$ contains three non-crossing paths linking v and q_2. Again, exactly two of these have odd length, and hence their union constitutes an ooe-Θ-graph. □

We note the following two consequences.

Lemma 21. *A t-perfect line graph can be coloured with at most* 3 *colours.*

Proof. This is a direct consequence of Lemmas 16 and 20. □

Next, we observe that the proof of Lemma 20 can easily be turned into a polynomial-time colouring algorithm for t-perfect line graphs. Indeed, given a t-perfect line graph H we first use Roussopoulos' [28] linear time algorithm to compute a graph G with $L(G) = H$. Then we iteratively colour the edges of G as outlined in Algorithm 1, where we use the notation of the proof of Lemma 20.

Lemma 22. *A t-perfect line graph can be coloured with three colours in polynomial time.*

We now prove the first of our two main results, which we restate.

Theorem 2. *Every claw-free t-perfect graph is* 3*-colourable.*

Algorithm 1 Edge colouring in ooe-Θ-free graphs

```
while there is an uncoloured edge e = uv:
    call colour(e)
output edge-colouring of G.

subroutine colour(e):
    if there is a colour c that is not used by the edges adjacent to e:
        Use c for e and return.
    Compute P.
    If v ∉ V(P):
        Swap colours on P.
        Use the free colour for e and return.
    Compute Q_1 and swap colours on Q_1.
    Compute Q_2 (which is possibly trivial) and swap colours on Q_2.
    Use the free colour for e and return.
```

For the proof we need a standard definition from the context of vertex-coloured graphs. The *Kempe-chain* in colours c_1, c_2 at a vertex v is the inclusion-maximal induced subgraph containing v that has vertices of colours c_1 and c_2 only. Recall that in a claw-free graph a Kempe-chain is always a path or a cycle.

Proof. Suppose otherwise, and let G be a counterexample with a minimum number of vertices. Hence $|V(G)| \geq 4$. As we may colour the blocks separately, and then combine their colourings to a colouring of G, we may assume that G is 2-connected.

First suppose that G is even 3-connected. Then, by Theorem 3, G does not contain any of K_4, W_5, C_7^2 or C_{10}^2 as a t-minor. Thus it follows from Lemma 9 that either G is a line graph, or $G \in \{C_6^2 - v_1v_6, C_7^2 - v_7, C_{10}^2 - v_{10}\}$. In the former case Lemma 21 implies that G is 3-colourable, while in the latter case the 3-colourability of G is easy to check. Thus in both cases we arrive at a contradiction.

Therefore we may assume that G is not 3-connected. Then there are vertices u, v which separate G. In other words, there are induced proper subgraphs L and R of G so that $V(L) \cap V(R) = \{u, v\}$ and $L \cup R = G$. As $|V(L)| < |V(G)|$ there is, by the choice of G, a 3-colouring c_L of L. Permuting colours, if necessary, we may assume that $c_L(u) = 1$ and $c_L(v) \in \{1, 2\}$.

Now in the graph L, consider the Kempe-chain K at v in colours $1, 2$. If $u \in V(K)$, then clearly, K contains an induced u–v path. On the other hand, if $u \notin V(K)$, then we can swap colours in K, thus obtaining a 3-colouring c'_L of L with $c_L(v) \neq c'_L(v)$. To sum up, we found that one of the following holds:

a) L contains an induced u–v path P using only colours 1 and 2 in c_L, or

b) L has 3-colourings c_i for $i = 1, 2$ so that $c_i(u) = 1$ and $c_i(v) = i$.

In case a), denote by $\tilde{R}$ the graph that is obtained from $R + P$ by performing t-contractions at inner vertices of P until either P has become an edge or u and v are identified. Clearly, $\tilde{R}$ is a proper t-minor of G, and hence, by minimality of G, there is a 3-colouring $c_{\tilde{R}}$ of $\tilde{R}$. The colouring $c_{\tilde{R}}$ naturally induces a 3-colouring c_R of R. Observe that by construction of $\tilde{R}$, we have $c_R(u) = c_R(v)$ if and only if $c_L(u) = c_L(v)$. So, by swapping colours if necessary we can combine c_L and c_R to a 3-colouring of G, which we assumed not to exist, a contradiction.

So we may assume that case b) above occurs. But then we may take any colouring of R (which exists by induction) and, swapping colours if necessary, combine it with the appropriate c_i to a 3-colouring of G, again a contradiction. □

With a slight modification, we can turn this proof into a colouring algorithm of polynomial time:

Lemma 23. *Every claw-free t-perfect graph on n vertices can be coloured with three colours in polynomial time in n.*

Algorithm 2 Colouring claw-free t-perfect graphs

```
output colour(G)

subroutine colour(G):
    Compute the blocks of G, call 2conncolour(B) for each block B,
    Swap colours in some blocks if necessary.
    Return the obtained 3-colouring of G.

subroutine 2conncolour(G):
    if G is 3-connected or |V(G)| < 4: return 3conncolour(G).
    Pick L ∪ R = G with V(L) ∩ V(R) = {u, v} and R ⊊ L so that R is minimal.
    Set c_L ← colour(L).
    Compute c_L(u), c_L(v)-Kempe chain K in L containing v.
    Set R' ← R̃ (the graph obtained from R by identifying u and v).
    if u ∈ V(K):
        if the u–v subpath in K has odd length: Set R' ← R + uv.
        call 3conncolour(R') to obtain a colouring c_R of R.            ①
        Permute colours in R so that c_L(u) = c_R(u) and c_L(v) = c_R(v).
        Combine c_L and c_R to a colouring c_G of G and return c_G.
    if the shortest u–v path in L has odd length: Set R' ← R + uv.
    call 3conncolour(R') to obtain a colouring c_R of R.                ②
    Permute colours in R so that c_L(u) = c_R(u).
    Change c_L to c'_L by swapping colours in K.
    Combine c_R with c_L or with c'_L to a colouring c_G of G and return c_G.

subroutine 3conncolour(G):
    if G ∈ {C_6^2 − v_1v_6, C_7^2 − v_7, C_10^2 − v_10}: return hardcoded 3-colouring.
    return a colouring of G obtained with Lemma 22.
```

Proof. Let us check that Algorithm 2 is correct. For the `subroutine` 3conncolour to work we clearly need that it is always applied to a 3-connected graph G, except when $|V(G)| < 4$. In the latter case, G is a line graph and so the reduction to Lemma 22 is valid (though unnecessary).

For the calls ① and ② we have to see that R' is 3-connected if $|V(R')| \geq 4$. Indeed, by the minimal choice of R this follows immediately when $R' = R + uv$. For $R' = \tilde{R}$, suppose that there are two vertices x, y that separate $\tilde{R}$. By the minimality of R, one of x and y, say y, must be the vertex we obtained by identifying u and v. Hence $\{u, v, x\}$ separates R. Since G is claw-free, the neighbourhood of u outside L lies in at most one component of $R - \{u, v, x\}$. Thus, also $\{v, x\}$ separates R, contradicting the minimality of R.

That the obtained colourings c_R and c_L indeed combine to 3-colourings of G follows from the arguments in the proof of Theorem 2.

Finally, to estimate the running time, note that `subroutine` 3conncolour needs only polynomial time, by Lemma 22. Moreover, every instruction, except the recursive call colour(L), can be performed in polynomial time. To determine L and R, for instance, we could consider all of the vertex sets of cardinality at most two. As $|V(L)| < |V(G)|$ holds in every step, the recursion depth is bounded by $|V(G)|$, which concludes the proof. □

6 Colouring h-perfect graphs

Let us now turn to h-perfect graphs, to which our results on colourings carry over. Sbihi and Uhry [29] introduced h-perfect graphs as a common generalisation of perfect and t-perfect graphs. For the definition of h-perfect graphs we use the same inequalities as for t-perfect graphs, only that the edge inequalities are replaced with clique inequalities. So, a graph is called *h-perfect* if the SSP is determined by

$$\begin{aligned}
&x \geq 0\\
&x(K) \leq 1 \text{ for every clique } K\\
&x(C) \leq \lfloor |V(C)|/2 \rfloor \text{ for every induced odd cycle } C.
\end{aligned}$$

Let us denote the fractional chromatic number by χ^*. More formally, if $\mathcal{S}$ denotes the set of all stable sets:

$$\begin{aligned}
\chi^*(G) \quad &= \quad \min \mathbf{1}^T y,\ y \in \mathbb{R}^{\mathcal{S}}\\
\text{subject to} \qquad & y \geq 0 \quad \text{and} \sum_{S \in \mathcal{S},\, v \in S} y_S \geq 1 \text{ for all } v \in V.
\end{aligned} \tag{13}$$

Define the polytope

$$P = \{x \in \mathbb{R}^V : x(S) \leq 1 \text{ for each stable set } S,\ x \geq 0\}.$$

Observe that $\max_{x \in P} \mathbf{1}^T x$ is the dual program of (13), so that we get $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$. Moreover, it is not hard to check that the anti-blocking polytope of P coincides with SSP(G). As G is h-perfect, Theorem 2.1 in Fulkerson [16] (see also [15]) yields therefore that every vertex $\neq \mathbf{0}$ of P is either the characteristic vector χ_K of a clique K of G or the vertex is of the form $\frac{2}{|C|-1}\chi_C$ for an odd cycle C.

In particular, for an h-perfect graph G we obtain

$$\chi^*(G) = \omega(G) \text{ if } \omega(G) \geq 3. \tag{14}$$

For claw-free graphs, we know more:

Theorem 24 (Sebő [26]). *Let G be a claw-free h-perfect graph. Then*

(i) $\chi(G) = \lceil \chi^*(G) \rceil$*; and*

(ii) $\chi(G) = \omega(G)$ *if* $\omega(G) \geq 3$.

The proof of this result uses our Theorem 2. All other arguments of the proof, however, have been developed by Sebő in order to show that Conjecture 1 on the 4-colourability of t-perfect graphs is implied by the following conjecture.

Conjecture 25 (Sebő [26]). *Every triangle-free t-perfect graph is 3-colourable.*

We can adapt this reduction to the claw-free case. As Sebő's argument has not been published we present it here.

Lemma 26 (Sebő [26]). *Let G be a claw-free graph with $\omega(G) \geq 3$. Then there exists a stable set that intersects every clique of size $\omega(G)$.*

Proof. Since $\omega(G) \geq 3 > \mathbf{1}^T(\frac{2}{|C|-1}\chi_C)$ for every odd cycle C of length ≥ 5, we see that $\max_{x \in P} \mathbf{1}^T x = \omega(G)$ is attained in every clique of size $\omega(G)$. Consider an optimal solution y of (13) and a clique K of size $\omega(G)$. Then

$$\omega(G) = \mathbf{1}^T \chi_K \leq \sum_S y_S \, \chi_S^T \chi_K = \sum_S y_S |S \cap K| \leq \sum_S y_S = \omega(G).$$

Thus, each stable set S with $y_S > 0$ must meet each such clique K. □

Proof of Theorem 24. Assume first that $\omega(G) \geq 3$. We find with Lemma 26 stable sets $S_1, \ldots, S_k$ where $k = \omega(G) - 3$ such that $G' := G - S_1 - \ldots - S_k$ has no clique of size 4. Thus, G' is t-perfect and therefore, by Theorem 2, 3-colourable. Hence G is $(k+3)$-colourable. This proves assertion (ii), and (i), too, for $\omega(G) \geq 3$ as $\omega(G)$ is a lower bound for $\chi^*(G)$.

Finally, assume $\omega(G) < 3$, that is, G is triangle-free. Then, since G is claw-free, it has maximum degree 2. So G is either a path or a hole for both of which (i) clearly holds. □

In the remainder of this section we demonstrate that a colouring as in Theorem 24 can be computed in polynomial time. Again we take inspiration from perfect graphs. Perfect graphs can be coloured in polynomial time, and we use as much of that proof as possible. It rests on two facts: in perfect graphs Max-Weight Stable Set is polynomial-solvable and, as a consequence of this and the weak perfect graph theorem, cliques of maximal size can be computed efficiently, too.

Max-Weight Stable Set. *Given a graph G and a weight $w : V(G) \to \mathbb{Z}_+$ compute a stable set S with maximal weight $w(S)$.*

Fortunately, Max-Weight Stable Set can be solved in polynomial time in h-perfect graphs, too.

Theorem 27 (Grötschel, Lovász and Schrijver [21]). *For h-perfect graphs* Max-Weight Stable Set *can be solved in polynomial time.*

The next result will help us in computing the clique number of a claw-free h-perfect graph.

Theorem 28 (Grötschel, Lovász and Schrijver [20]). *For any collection $\mathcal{G}$ of graphs holds: There is a polynomial-time algorithm to find a fractional weighted colouring number for any graph in $\mathcal{G}$ if and only if there is a polynomial-time algorithm that solves* Max-Weight Stable Set *for any graph in $\mathcal{G}$ and any integer weight function.*

Lemma 29. *For an h-perfect graph G it is possible to compute $\omega(G)$, and then also a clique of size $\omega(G)$, in polynomial time.*

Proof. First, check whether G contains a triangle. If not, then it is easy to determine $\omega(G)$. So assume that $\omega(G) \geq 3$, and observe that as a direct consequence of Theorems 27 and 28 we can calculate $\chi^*(G)$ in polynomial time. Since $\chi^*(G)$ and $\omega(G)$ coincide, by (14), we have determined $\omega(G)$.

Now, a standard argument (see for instance [30, Corollary 67.2b]) allows us to find a clique of size $\omega(G)$: Pick a vertex v and use the algorithm above to check whether $\omega(G) > \omega(G - v)$. If no, replace G by $G - v$ and repeat. If yes, pick the next vertex until $\omega(G) > \omega(G - v)$ for all vertices v, in which case the remaining graph forms a clique of size $\omega(G)$. □

Theorem 30.[2] *For h-perfect claw-free graphs an optimal colouring can be computed in polynomial time.*

Proof. In view of Theorem 23 and the proof of Theorem 24 we only need to show that a stable set that intersects every clique of maximal size can be found in polynomial time—provided that $\omega(G) \geq 3$. To do so, we follow arguments given by Grötschel, Lovász and Schrijver [22] for an analogous result for perfect graphs; see also [30, Corollary 67.2c].

Algorithm 3 Computing the desired stable set

Use Lemma 29 to compute a clique K of G with maximal size.
Set $t \leftarrow 0$ and $\omega \leftarrow |K|$.
`while` $\omega = |K|$:
    Set $K_{t+1} \leftarrow K$ and $t \leftarrow t + 1$.
    Use the algorithm of Theorem 27 to compute a stable set S of G with maximal weight with respect to the weight $w := \chi_{K_1} + \dots \chi_{K_t}$.
    Use Lemma 29 to compute a clique K of $G - S$ with maximal size.
`output` S.

We use Algorithm 3 to compute the desired stable set. Obviously, if the number of iterations is bounded by a polynomial in $|V(G)|$ then the theorem follows. We note that by Lemma 26 any S as computed in the algorithm intersects each of $K_1, \dots, K_t$. This means that χ_S is contained in $L_t := \{x \in \mathbb{R}^V : x(K_i) = 1 \text{ for } i = 1, \dots, t\}$. However, the dimension of L_t drops with every t since the χ_{K_i} are linearly independent. Thus, the number of iterations is bounded by $|V(G)|$. □

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References

[1] F. Barahona and A. R. Mahjoub, *Compositions of graphs and polyhedra II: stable sets*, SIAM J. Discrete Math. **7** (1994), 359–371.

[2] M. Boulala and J.P. Uhry, *Polytope des indépendants d'un graphe série-parallèle*, Disc. Math. **27** (1979), 225–243.

[3] D. Cao and G.L. Nemhauser, *Polyhedral characterizations and perfection of line graphs*, Disc. Appl. Maths. **81** (1998), 141–154.

[4] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vuskovic, *Recognizing berge graphs*, Combinatorica **25** (2005), 143–187.

[5] M. Chudnovsky and A. Ovetsky, *Coloring quasi-line graphs*, J. Combin. Theory (Series B) **54** (2007), 41–50.

[6] M. Chudnovsky and P. Seymour, *The structure of claw-free graphs*, Surveys in combinatorics 2005, vol. 327, London Math Soc Lecture Note, 2005, pp. 153–171.

[7] M. Chudnovsky, R. Seymour, N. Robertson, and R. Thomas, *The strong perfect graph theorem*, Ann. Math. **164** (2006), 51–229.

[8] V. Chvátal, *On certain polytopes associated with graphs*, J. Combin. Theory (Series B) **18** (1975), 138–154.

[9] G. Dahl, *Stable set polytopes for a class of circulant graphs*, SIAM J. Optim. **9** (1999), 493–503.

[10] R. Diestel, *Graph theory* (3rd edition), Springer-Verlag, 2005.

[11] J. Edmonds, *Maximum matching and a polyhedron with* $0,1$*-vertices*, J. Res. Natl. Bur. Stand., Sect. B **69** (1965), 125–130.

[12] F. Eisenbrand, S. Funke, N. Garg, and J. Könemann, *A combinatorial algorithm for computing a maximum independent set in a t-perfect graph*, In Proc. of the ACM-SIAM Symp. on Discrete Algorithms, ACM, 2002, pp. 517–522.

[13] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura, *The stable set polytope of quasi-line graphs*, Combinatorica **28** (2008), 45–67.

[14] J. Fonlupt and J.P. Uhry, *Transformations which preserve perfectness and h-perfectness of graphs*, Ann. Disc. Math. **16** (1982), 83–95.

[15] D.R. Fulkerson, *Blocking and anti-blocking pairs of polyhedra*, Math. Programming **1** (1971), 168–194.

[16] ______, *Anti-blocking polyhedra*, J. Combin. Theory (Series B) **12** (1972), 50–71.

[17] A. Galluccio and A. Sassano, *The rank facets of the stable set polytope for claw-free graphs*, J. Combin. Theory (Series B) **69** (1997), 1–38.

[18] A.M.H. Gerards, *Graphs and polyhedra - binary spaces and cutting planes*, CWI Tract 73, CWI, Amsterdam, 1990.

[19] A.M.H. Gerards and F.B. Shepherd, *The graphs with all subgraphs t-perfect*, SIAM J. Discrete Math. **11** (1998), 524–545.

[20] M. Grötschel, L. Lovász, and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), 169–197.

[21] ______, *Relaxations of vertex packing*, J. Combin. Theory (Series B) **40** (1986), 330–343.

[22] ______, *Geometric algorithms and combinatorial optimization*, Springer-Verlag, 1988.

[23] F. Harary, *Graph theory*, Addison Wesley, 1972.

[24] L. Lovász, *A note on factor-critical graphs*, Studia Scientiarum Mathematicum Hungarica **7** (1972), 279–280.

[25] A. R. Mahjoub, *On the stable set polytope of a series-parallel graph*, Math. Programming **40** (1988), 53–57.

[26] A. Sebő, personal communication.

[27] W. Pulleyblank and J. Edmonds, *Facets of* 1*-matching polyhedra*, Hypergraph Seminar (C. Berge and D. Ray-Chaudhuri, eds.), Springer, 1974, pp. 214–242.

[28] N.D. Roussopoulos, *A* $max\{m,n\}$ *algorithm for determining the graph* H *from its line graph* G, Inf. Process. Lett. **2** (1973), 108–112.

[29] N. Sbihi and J.P. Uhry, *A class of h-perfect graphs*, Disc. Math. **51** (1984), 191–205.

[30] A. Schrijver, *Combinatorial optimization. Polyhedra and efficiency*, Springer-Verlag, 2003.

[31] F. B. Shepherd, *Applying Lehman's theorems to packing problems*, Math. Prog. **71** (1995), 353–367.

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