Appendix

The lemma of Barahona and Mahjoub

Our Lemma 19 is a slight reformulation of Theorem 2.5 of Barahona and Mahjoub [1]. Fortunately, only minor modifications are necessary to adapt the proof. For the sake of completeness the adapted proof follows. We stress that it is mostly the original work of Barahona and Mahjoub.

Lemma 1 (Barahona and Mahjoub [1])**.** *Let G be a graph, and let uvw be a path in G with* $\deg(v) = 2$ *so that $uw \notin E(G)$. Furthermore, let $a^T x \leq \alpha$ be a facet-defining inequality of SSP(G) so that $a_u = a_v = a_w > 0$. Denote by G' the graph obtained from G by contracting uv and vw, and let $\tilde{v}$ be the new vertex, i.e. $V(G') \setminus V(G) = \{\tilde{v}\}$. If $a' \in \mathbb{R}^{V(G')}$ is defined by $a'_p = a_p$ for $p \in V(G' - \tilde{v})$ and $a_{\tilde{v}} = a_v$ then $a'^T x \leq \alpha - a_v$ is a facet-defining inequality of SSP(G').*

Proof. First, let us see that $a'^T x \leq \alpha - a_v$ is valid for $\mathrm{SSP}(G')$. For this, let S' be a stable set of G'. If $\tilde{v} \in S'$ then $S := (S' \setminus \tilde{v}) \cup \{u, w\}$ is a stable set of G. Thus $\alpha - a_v \geq a^T \chi_S - a_v = a'^T \chi_{S'}$. If, on the other hand, $\tilde{v} \notin S'$ then $S := S' \cup \{v\}$ is stable in G, and we get again $\alpha - a_v \geq a^T \chi_S - a_v = a'^T \chi_{S'}$.

Second we show that $a'^T x \leq \alpha - a_v$ is a facet. For later use, we note that

$$\alpha \geq 2a_v. \tag{1}$$

Indeed, as $uw \notin E(G)$, the vector $\chi_{\{u,w\}}$ lies in $\mathrm{SSP}(G)$, and hence $\alpha \geq a^T \chi_{\{u,w\}} = 2a_v$.

Since $a^T x \leq \alpha$ defines a facet of $\mathrm{SSP}(G)$ there are $n = |V(G)|$ affinely independent stable sets $S_1, \ldots, S_n$ of G that satisfy $a^T x \leq \alpha$ with equality. As $\alpha \neq 0$ the stable sets are even linearly independent. For each i if $\{u, w\} \subseteq S_i$ we set $S'_i := S_i \setminus \{u, v, w\} \cup \{\tilde{v}\}$. Otherwise, we define S'_i to be $S_i \setminus \{u, v, w\}$.

Clearly, the S'_i are stable sets in G' that satisfy $a'^T x \leq \alpha - a_v$ with equality. If we can show that they form an affinely independent set then we are done. Denote by M the matrix whose columns are χ_{S_i}, and let M' be the matrix with columns $\chi_{S'_i}$. Then the matrices have the following form:

$$M = \begin{array}{cc} \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 1\ldots1 & 1\ldots1 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & 0\ldots0 & 1\ldots1 \\ 1\ldots1 & 0\ldots0 & 1\ldots1 & 0\ldots0 \end{pmatrix} & \begin{matrix} \\ \leftarrow u \\ \leftarrow v \\ \leftarrow w \end{matrix} \end{array}$$

$$M' = \begin{array}{cc} \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 1\ldots1 & 0\ldots0 & 0\ldots0 & 0\ldots0 \end{pmatrix} & \begin{matrix} \\ \leftarrow \tilde{v} \end{matrix} \end{array}$$

We show that M' has rank $n - 2$. For this we consider a third matrix $\bar{M}$:

$$\bar{M} = \begin{array}{cc} \begin{pmatrix} & 0 \\ & \vdots \\ M & 0 \\ & 1 \\ & 0 \\ & 0 \\ 1\ldots1 & 1 \end{pmatrix} & \begin{matrix} \\ \\ \\ \leftarrow u \\ \leftarrow v \\ \leftarrow w \\ \\ \end{matrix} \end{array}$$

Suppose that $\bar{M}$ is singular. As M has full rank, this implies that the last row of $\bar{M}$ is a linear combination of the other rows. As the only solution of $x^T M = (\alpha \dots \alpha)$ is a it follows that $a_u = \alpha$, a contradiction to $a_u = a_v > 0$ and (1). Therefore, $\bar{M}$ is non-singular.

Finally, we add the rows corresponding to v and w to the one corresponding to u and substract from the resulting row the last row of $\bar{M}$. We get:

$$\begin{array}{c} \left(\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & 0 \\ 1\dots1 & 0\dots0 & 0\dots0 & 0\dots0 & 0 \\ 0\dots0 & 0\dots0 & 0\dots0 & 1\dots1 & 0 \\ 1\dots1 & 0\dots0 & 1\dots1 & 0\dots0 & 0 \\ 1\dots1 & 1\dots1 & 1\dots1 & 1\dots1 & 1 \end{array}\right) \begin{array}{c} \\ \leftarrow u \\ \leftarrow v \\ \leftarrow w \\ \\ \end{array} \end{array}$$

Since this matrix is non-singular and contains M' as a submatrix we deduce that M' has rank $n-2$, as desired. □

Sebő's corollary

We give here a slightly more comprehensive proof of Corollary 16. Recall that a graph is called *h-perfect* if its SSP is determined by

$$\begin{aligned} & x \geq 0 \\ & x_K \leq 1 \text{ for every clique } K \\ & x(C) \leq \lfloor |C|/2 \rfloor \text{ for every induced odd cycle } C \end{aligned}$$

We denote by HSTAB(G) the polytope determined by those three types of inequalities.

The proof of the following corollary is due to Sebő [5]. As it has not been published but contains a nice and useful technique we present it here.

Corollary 1. *Let G be a claw-free h-perfect graph. Then*

(i) $\chi(G) = \lceil \chi^*(G) \rceil$*; and*

(ii) $\chi(G) = \omega(G)$ *if* $\omega(G) \geq 3$.

Here, χ^* denotes the fractional chromatic number. More formally, if $\mathcal{S}$ denotes the set of all stable sets:

$$\begin{array}{rcl} \chi^*(G) & = & \min \mathbf{1}^T y,\ y \in \mathbb{R}^{\mathcal{S}} \\ \text{subject to} & & y \geq 0 \\ & & \displaystyle\sum_{S \in \mathcal{S},\, v \in S} y_S \geq 1 \text{ for all } v \in V \end{array} \tag{2}$$

To state the dual of this linear program, define the polytope

$$P = \{x \in \mathbb{R}^V : x(S) \leq 1 \text{ for each stable set } S,\ x \geq 0\}.$$

Then, by duality, we have $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$.

Let $Q \subseteq \mathbb{R}^n_+$ be a polytope. Define

$$Q^{\downarrow} := \{y \in \mathbb{R}^n : \text{ there is } x \in Q \text{ with } y \leq x\}.$$

We say that Q is of the *anti-blocking type* if $Q = Q^{\downarrow} \cap \mathbb{R}^n_+$. The *anti-blocking polytope* $A(Q)$ of Q is then defined to be

$$A(Q) := \{z \in \mathbb{R}^n_+ : x^T z \leq 1 \text{ for all } x \in Q\}.$$

For a set $S \subseteq \mathbb{R}^V$ we denote by χ_S the characteristic vector of S. As they are used in different context we hope that the danger of confusing the chromatic number χ with a characteristic vector χ_S is minimal.

Theorem 2 (Fulkerson [3] and [4])**.** *Let $Q \subseteq \mathbb{R}^n_+$ be a full-dimensional polytope of the anti-blocking type, and let $a_1, \ldots, a_k \in \mathbb{R}^n_+$. Then $Q = \mathrm{conv}\{a_1, \ldots, a_k\}^{\downarrow} \cap \mathbb{R}^n_+$ if and only if $A(Q) = \{z \in \mathbb{R}^n_+ : a_i^T z \leq 1$ for $i = 1, \ldots, k\}$.*

For an h-perfect G and P as above it holds that

$$\mathrm{SSP}(G) = A(P).$$

Indeed, consider a $z \in \mathrm{SSP}(G)$. Now, since for any $x \in P$ we have $x^T \chi_S \leq 1$ for all $S \in \mathcal{S}$, the same inequality holds for z, as it is a convex combination of characteristic vectors of stable sets. Hence, $z \in A(P)$.

Conversely, let $z \in A(P)$. Observe that for a clique K, and an odd cycle C the vectors χ_K and $\frac{2}{|C|-1}\chi_C$ lie in P. Thus, $z^T \chi_K \leq 1$ and $z^T \frac{2}{|C|-1}\chi_C \leq 1$, which implies $z \in \mathrm{HSTAB}(G) = \mathrm{SSP}(G)$.

Therefore, Fulkerson's theorem has the following consequence:

$$P = \mathrm{conv}\{\chi_{K_1}, \ldots, \chi_{K_s}, \tfrac{2}{|C_1|-1}\chi_{C_1}, \ldots, \tfrac{2}{|C_t|-1}\chi_{C_t}\}^{\downarrow} \cap \mathbb{R}^n_+, \tag{3}$$
where $\{K_1, \ldots, K_s\}$ is the set of all cliques and $\{C_1, \ldots, C_t\}$ is the set of all odd cycles of G.

Proof of Corollary 1. Let P be defined as above.

First, assume that $\omega(G) \geq 3$. We show that

there is a stable set S which intersects every clique of size $\omega(G)$. (4)

Since, $\omega(G) \geq 3 > \mathbf{1}^T(\frac{2}{|C|-1}\chi_C)$ for every odd cycle C of length ≥ 5, we see with (3) that $\max_{x \in P} \mathbf{1}^T x = \omega(G)$ is attained in every clique of size $\omega(G)$. Consider an optimal solution y of (2) and a clique K of size $\omega(G)$. Then

$$\omega(G) = \mathbf{1}^T \chi_K \leq \sum_S y_S\, \chi_S^T \chi_K = \sum_S y_S |S \cap K| \leq \sum_S y_S = \omega(G).$$

Thus, each stable set S with $y_S > 0$ must meet each such clique K, which proves (4).

Next, we find with (4) stable sets $S_1, \ldots, S_k$ where $k = \omega(G) - 3$ such that $G' := G - S_1 - \ldots - S_k$ has no clique of size 4. Thus, G' is t-perfect and therefore, by Theorem 2 of [2], colourable with three stable sets, $S_{k+1}, S_{k+2}, S_{k+3}$ say. Now, we can colour G with $S_1, \ldots, S_{\omega(G)}$. This proves assertion (ii), and (i), too, for $\omega(G) \geq 3$ as $\omega(G)$ is a lower bound for $\chi^*(G)$.

Finally, assume $\omega(G) < 3$. If G is not bipartite, in which case we are done, then $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$ is attained in $\frac{2}{|C|-1}\chi_C$ for some odd cycle C. Thus, $\chi^*(G) > 2$. On the other hand, G is t-perfect, and we can consequently, by Theorem 2 of [2], colour it with three colours. ☐

References

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