

h -perfect plane triangulations

Yohann Benchetrit and Henning Bruhn

Abstract

We characterise t -perfect plane triangulations by forbidden induced subgraphs. As a consequence, we obtain that a plane triangulation is h -perfect if and only if it is perfect.

1 Introduction

As defined by Berge, a graph is *perfect* if for each of its induced subgraphs, the chromatic number is equal to the clique number. Results of Fulkerson [7], Lovász [11] and Chvátal [4] showed that perfection could equally be characterised in polyhedral terms: a graph is perfect if and only its stable set polytope (the convex hull of its stable sets) is fully described by non-negativity and clique constraints (we defer precise definitions to the next section).

The polyhedral setting for perfection suggested several generalisations. A graph is *h -perfect* if its stable set polytope is defined by non-negativity, clique and odd-cycle constraints; the K_4 -free h -perfect graphs are called *t -perfect*.

In this note, we show that h -perfection and perfection are equivalent for plane triangulations.

Theorem 1. *A plane triangulation is h -perfect if and only if it is perfect.*

To our knowledge, this is the first example of a rather wide class for which these two notions coincide. The equivalence is a consequence of a characterisation of t -perfect plane triangulations, see Theorem 2 below. This in itself is remarkable, as a characterisation of h -perfection in a class of graphs does not normally follow directly from one for t -perfection.

The Strong Perfect Graph theorem [3] gives a structural characterisation of perfect graphs in terms of minimal imperfect obstructions: a graph is perfect if and only if it contains neither an odd hole (an induced odd cycle other than a triangle) nor the complement of an odd hole. An analogous characterisation for t -perfect or h -perfect graphs is not known. Our second main result is such a characterisation of t -perfect plane triangulations.

Which (minimal) t -imperfect graphs could occur in a plane triangulation? Odd wheels form a family of minimally t -imperfect graphs (see for instance Schrijver [12, Ch. 68]), and they are planar. Consequently, a planar t -perfect graph cannot contain any odd wheel as an induced subgraph (t -perfection is closed under taking induced subgraphs). Excluding odd wheels, however, is not enough to ensure t -perfection, even for plane triangulations (see Figure 1). Rather, we have to exclude certain subdivisions of odd wheels as well.

For this, we call a *loose odd wheel* any graph that is obtained from an odd wheel by a subdivision of the edges of the rim, where the number of edges that

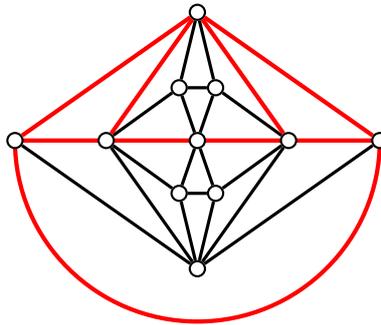


Figure 1: A plane triangulation that is t -imperfect but that does not contain any induced odd wheel. A loose odd wheel is shown in red.

are subdivided an even number of times is odd, and at least three. (Again, more precise definitions follow in the next section.) We prove:

Theorem 2. *For every plane triangulation T , the following statements are equivalent:*

- (i) T is t -perfect;
- (ii) T does not contain any loose odd wheel as an induced subgraph; and
- (iii) T is perfect and K_4 -free.

Combining the two theorems, it is straightforward to show that a plane triangulation is h -perfect if and only if it does not contain any loose odd wheel other than K_4 as an induced subgraph.

The h -perfect graphs have received some attention due to their algorithmic properties, which they share to some degree with perfect graphs. In particular, Grötschel, Lovász and Schrijver [10] showed that a maximum-weight stable set of an h -perfect graph can be found in polynomial-time using the Ellipsoid method. Later Eisenbrand et al. [6] described a combinatorial algorithm for t -perfect graphs. The class of t -perfect graphs is of independent interest as it is the class of graphs whose fractional stable set polytope (that is the polytope defined by non-negativity and edge inequalities) has Chvátal rank at most 1 (see [12]).

Key questions about t -perfect or h -perfect graphs are: How can they be described in terms of forbidden substructures? Can they always be coloured with few colours? Is it possible to recognise them in polynomial time? These questions have been pursued in a number of works, among them [9, 13, 8, 2, 1]. See also Schrijver [12, Ch. 68] for more references.

In the special case of plane triangulations, Theorem 2 answers the first question and Theorem 1 immediately answers the other ones: perfect graphs can always be coloured with no more colours than the size of a maximum clique, and they can be recognised in polynomial-time (the planar case is considerably simpler; see Tucker [14]).

2 Basic definitions and facts

We keep to notation as in Diestel [5], where also all general definitions that we omit here may be found. We only consider graphs that are undirected, simple and finite.

A graph is *perfect* if for each induced subgraph, the chromatic number and clique number are equal. For each graph G , consider the following inequalities over $\mathbb{R}^{V(G)}$:

$$x_v \geq 0 \text{ for every } v \in V(G) \quad (1)$$

$$x_u + x_v \leq 1 \text{ for every edge } uv \in E(G) \quad (2)$$

$$\sum_{v \in K} x_v \leq 1 \text{ for every clique } K \text{ of } G \quad (3)$$

$$\sum_{v \in V(C)} x_v \leq \frac{|V(C)| - 1}{2} \text{ for every induced odd cycle } C \text{ in } G \quad (4)$$

The graph G is perfect if and only if the polytope described by (1) and (3) is integral (that is, if all the vertices have integer coordinates only).

A graph is *h -perfect* if the polytope described by the inequalities (1), (3) and (4) is integral. This is precisely the case when the polytope is the convex hull of the incidence vectors of the stable sets of G . A graph is *t -perfect* if (1), (2) and (4) define an integral polytope.

Perfect graphs are *h -perfect* (since they cannot contain induced odd cycles of length greater than 3), and it is straightforward to show that *t -perfect* graphs are exactly the *h -perfect* K_4 -free graphs.

As for perfect graphs, each induced subgraph of an *h -perfect* graph is *h -perfect* (this follows from an easy polyhedral argument), and hence the same holds for *t -perfection*.

A *hole* is an induced cycle of length at least 4; an *antihole* is the complement of a hole. Odd holes and odd antiholes are imperfect, and indeed the Strong Perfect Graph theorem [3] asserts that these are the only *minimally* imperfect graphs (in this note, *minimally* is with respect to deletion of vertices): a graph is perfect if and only if it does not contain an odd hole nor an odd antihole (as an induced subgraph). A similar structural characterisation of *t -perfection* or *h -perfection* is not known.

We will use twice a well-known lemma in the context of perfect graphs. Its proof is straightforward from Berge's original definition of perfection.

Lemma 3 (Chvátal [4]). *Let G be a graph and G_1, G_2 be two proper induced subgraphs of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique. Then G is perfect if and only if G_1 and G_2 are perfect.*

A *wheel* is a graph formed by a cycle C (the *rim*) together with a vertex v (the *center*) adjacent to every vertex of C . A wheel is *odd* if it has an odd rim.

It is easy to show that odd wheels are minimally *t -imperfect*; see for instance Schrijver [12, Ch. 68]. Other minimally *t -imperfect* graphs are even Möbius ladders, and the circular graph C_{10}^2 ; see Figure 5. Only odd wheels will be of relevance in this note.

A *loose wheel* is a graph formed by a cycle C and a new vertex $v \notin V(C)$ that has at least three neighbours on C . A path of C joining two neighbours

of v and which does not contain any other neighbour of v is a *segment* of the loose wheel. The loose wheel is a *loose odd wheel* if C has odd length, and if at least three of the segments are odd as well. Obviously, odd wheels are loose odd wheels.

To see that the presence of a loose odd wheel in the triangulation actually certifies t -imperfection, we turn to a useful operation, found by Gerards and Shepherd [8], that preserves t -perfection. In a graph G , let v be a vertex whose neighbourhood is stable (i.e., that does not induce any edge). Then, performing a t -contraction at v means contracting all the edges incident with v (deleting parallel edges and loops which may arise). Results of [8] show that the resulting graph G' will be t -perfect if G is. This implies:

Lemma 4. *Each graph which contains a loose odd wheel as an induced subgraph is t -imperfect.*

Proof. Since t -perfection is closed under taking induced subgraphs, we need only to check that a loose odd wheel is t -imperfect. We show that it can be t -contracted to an odd wheel, which is t -imperfect.

The number k of odd segments of a loose odd wheel is odd and at least 3. We may use t -contraction to shrink each even segment to a single vertex and each odd segment to a single edge. It is easy to check that the graph obtained in this way is isomorphic to the odd wheel with $k + 1$ vertices. \square

3 Proofs

Let us first show that Theorem 1 is a straightforward consequence of Theorem 2.

Proof of Theorem 1. Perfect graphs are always h -perfect (see Section 2).

Conversely, let T be an h -perfect plane triangulation. If T is K_4 -free, then T is t -perfect and the conclusion follows from Theorem 2. Otherwise, T is either equal to K_4 , in which case it is perfect, or one of the triangles X of any K_4 of T is separating. Thus, X yields two proper induced subgraphs as in Lemma 3 and we are done by induction. \square

We will use the following observation:

Lemma 5. *A plane triangulation without separating odd holes is perfect.*

Proof. Such a triangulation T does not contain any odd hole at all, as every face boundary is a triangle. Moreover, T does not contain any odd antihole either: the only planar odd antihole is the antihole with five vertices, which is isomorphic to the cycle of length 5. Therefore, T is perfect (this follows from the Strong Perfect Graph Theorem, in the easier special case of planar [14] or K_4 -free graphs [15]). \square

The key ingredient in the proof of Theorem 2 is the following:

Lemma 6. *Any plane triangulation that has an odd hole contains either an induced loose odd wheel or a separating triangle.*

We now use this to prove Theorem 2:

Proof of Theorem 2. (i) \Rightarrow (ii) follows directly from Lemma 4.

To show (ii) \Rightarrow (iii), suppose that T is a minimal counterexample, that is, a plane triangulation without any induced loose odd wheel, which is not perfect and with $|V(T)|$ minimum. Since K_4 is an odd wheel, T is K_4 -free. By Lemma 5, T must have a separating odd hole, and thus, by Lemma 6, also a separating triangle X . Clearly, X yields two proper induced subgraphs as in Lemma 3, which are perfect. But then T is perfect, contrary to our assumption.

Finally, (iii) \Rightarrow (i) follows from the polyhedral characterisation of perfect graphs and the obvious fact that perfect graphs cannot contain odd holes (see Section 2). \square

We now prove Lemma 6. The *interior* of a cycle C of a plane graph is the bounded component of $\mathbb{R}^2 \setminus C$.

Proof of Lemma 6. Let T be a plane triangulation that contains an odd hole but no separating triangle. We need to show that T contains an induced loose odd wheel. Let C be an odd hole of T with a minimal number of vertices in the interior of C . We claim the following:

Some vertex in the interior of C has at least three neighbours in C . (5)

We first show how to deduce that T contains an induced loose odd wheel from (5). Let v be a vertex in the interior of C with at least three neighbours in C , and denote by W the induced loose wheel formed by v and C . If W is a loose odd wheel, we are obviously done. So, assume that W has fewer than three odd segments. Since C is odd, this means that W has exactly one odd segment, P say.

Let v_1 and v_2 be the two endvertices of P . For $i \in \{1, 2\}$, there is exactly one segment P_i which has v_i as an endvertex and is not P . Let v'_i be the other endvertex of P_i (note that v'_1 and v'_2 are identical if v has exactly three neighbors on C).

Let $i \in \{1, 2\}$ and let D_i be the cycle formed by vv_i , vv'_i and P_i . Let F be the face boundary of the face of $T - v$ containing v . Note that v is adjacent to every vertex of F , as T is a triangulation, and that v_i and v'_i lie in F . In particular, F contains a path Q_i from v_i to v'_i so that all its inner vertices lie in the interior of D_i . Observe that Q_i is induced: a chord xy of Q_i would yield a separating triangle xyv , which we had excluded. Since v_i and v'_i are not neighbours (otherwise P_i would not have even length), it follows that the induced path Q_i has length at least 2. (Here, we also use that as a segment of the hole C , the path P_i is induced.)

Now, if Q_i is even then Q_i and the odd path of C joining v_i to v'_i form an odd hole of T with fewer vertices in the interior than C , contradicting the minimality of C .

Hence, both Q_1 and Q_2 must be odd. Since P is the unique odd segment of W , this implies that: Q_1 , Q_2 , the even path of C joining v'_1 to v'_2 and P form together an odd hole of T . We can see directly that the odd hole together with v forms a loose odd wheel, and we are done.

This shows that the lemma can be deduced from (5). All that remains is to prove (5). For this purpose, suppose (5) to be false. That is, suppose that every vertex in the interior of C has at most two neighbours on C .

Let $N = |V(C)|$ and let v_1, \dots, v_N be a circular ordering of the vertices of C . In what follows, indices are always taken modulo N .

Since T is a triangulation, the vertices v_i and v_{i+1} have a common neighbour w_i in the interior of C (for each $i \in \{1, \dots, N\}$). By assumption, w_i has no other neighbour in C besides v_i and v_{i+1} .

Observe that:

no vertex in the interior of C , except for w_1, \dots, w_N , has two neighbours in C . (6)

Indeed, if u is another vertex in the interior of C with at least (and then exactly) two neighbors v_i and v_j on C , then v_i and v_j cannot be consecutive on C (because T does not have a separating triangle). Therefore, the edges uv_i , uv_j and the odd path of C joining v_i to v_j form an odd hole with fewer vertices in the interior than C , which is impossible.

Next, let us see that:

the neighbours of v_{i+1} in the interior of C form an induced path R_i from w_i to w_{i+1} , for every $i \in \{1, \dots, N\}$. (7)

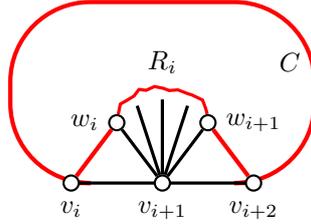


Figure 2: The path R_i and the cycle C' (in red)

Indeed, consider the face of $T - v_{i+1}$ containing v_{i+1} . Its face boundary consists of neighbours of v_{i+1} only (as T is a triangulation), and contains a path R'_i from v_i to v_{i+2} whose inner vertices lie in the interior of C . Moreover, R'_i is induced since any chord xy would yield a separating triangle xyv_{i+1} .

Since $w_i v_i$ and $w_{i+1} v_{i+2}$ are edges of T , w_i succeeds v_i and w_{i+1} precedes v_{i+2} in R'_i . Choose R_i as the subpath of R'_i from w_i to w_{i+1} (see Figure 2). This proves (7).

Let us note rightaway that for each $i \in \{1, \dots, N\}$:

inner vertices of R_i have no other neighbour in C than v_{i+1} . (8)

Suppose that r is an inner vertex of R_i that is adjacent to $v_j \neq v_{i+1}$. Then r has two neighbours in C and thus, by (6), has to be one of w_1, \dots, w_N . Since for every $l \in \{1, \dots, N\}$, w_l has exactly v_l and v_{l+1} as neighbours in C , it follows that $r = w_i$ or $r = w_{i+1}$. This is impossible since r is an inner vertex of R_i .

Next:

each R_i has odd length. (9)

If R_i was even then $C - v_{i+1}$, R_i and the edges $v_i w_i$ and $w_{i+1} v_{i+2}$ would form together an odd hole C' ((8) shows that C' is induced). Clearly, C' has fewer vertices in its interior than C , and this contradicts the minimality of C ; see Figure 2.

We need one more fact. For each $i, j \in \{1, \dots, N\}$:

$$\text{if } w_i w_j \in E(T) \text{ then } i = j + 1 \text{ or } j = i + 1. \quad (10)$$

Suppose that $i \notin \{j-1, j+1\}$. Then, C contains two disjoint paths, one between v_{i+1} and v_j , and the other between v_{j+1} and v_i . One of these paths has even length, and in particular length at least 2; denote the path by P . We extend P by the two edges between its endvertices and w_i and w_j , and finally add the edge $w_i w_j$. The resulting cycle is induced and of odd length at least 5, which is impossible as it contradicts the minimal choice of C .

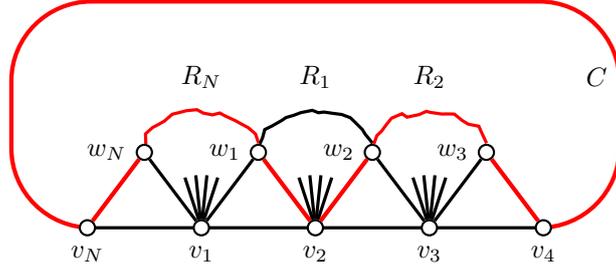


Figure 3: The cycle C_1 (in red)

Now, consider the walk $R := w_1 R_1 w_2 R_2 w_3 \dots w_N R_N w_1$. By (7) and (8), R is a cycle of length at least 5. By (9), it is odd. Since there are fewer vertices in the interior of R than in the interior of C , the minimality of C ensures that R has a chord.

As every R_i is induced (by (7)) and as vertices w_i and w_j may be adjacent only if $|i - j| = 1$ (as stated by (10)), such a chord must have at least one endvertex that is an inner vertex of some R_i , say R_1 . In particular, R_1 is not an edge.

Let C_1 be the cycle formed by $v_2 w_2$, R_2 , $w_3 v_4$, $v_N w_N$, R_N , $w_1 v_2$ and the odd $v_4 - v_N$ path of C (see Figure 3). By (9), C_1 has odd length. Obviously, C_1 is not a triangle and has fewer vertices in its interior than C . Hence, it must have a chord xy .

By (8) and as the R_i are induced, this is only possible if one of x and y , say x , lies in R_2 while y lies in R_N . Furthermore, xy lies in the interior of R (except its ends).

It is easy to check that $\{x, y\} \cap \{w_1, w_2\} \neq \emptyset$: otherwise, $v_1 x y v_3 v_2 v_1$ is an odd hole (of length 5), contradicting the minimality of C .

Therefore, and by symmetry, we may assume without loss of generality that $x = w_2$. Since R_1 is induced and not an edge, this implies $y \neq w_1$. For later use, we note the following trivial consequence (see Figure 4):

$$\text{the interior of the cycle } D = v_1 v_2 w_2 y v_1 \text{ contains } R_1 - w_2 \text{ but} \quad (11) \\ \text{no vertex of } C.$$

Observe moreover that $y \neq w_N$ by (10). In particular, this means that y is an inner vertex of R_N , which thus cannot be an edge either. Let C_N be the odd cycle formed by $v_1 w_1$, R_1 , $w_2 v_3$, $v_{N-1} w_{N-1}$, R_{N-1} , $w_N v_1$, and the odd $v_3 - v_{N-1}$ path of C . That is, we construct C_N in the exact way as C_1 , only for R_N instead of R_1 .

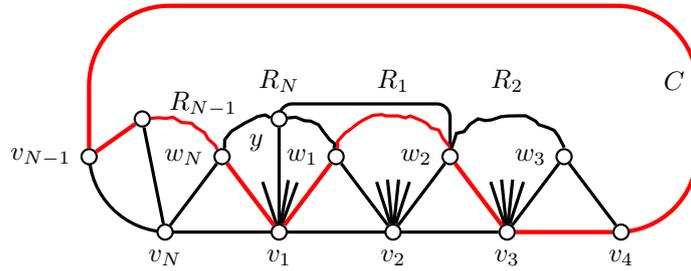


Figure 4: The cycle C_N (in red)

As for C_1 , the cycle C_N has a chord that joins a vertex s of R_{N-1} to a vertex t of R_1 . Moreover, we deduce in the same way as for C_1 that $\{s, t\} \cap \{w_1, w_N\} \neq \emptyset$.

Since we can reach C from s without meeting the cycle D (using the path $R_{N-1}w_{N-1}v_{N-1}$), it follows that s does not lie in the interior of D . On the other hand $R_1 - w_2$ lies in the interior of D (see (11)), thus $t = w_2$. As $\{s, t\} \cap \{w_1, w_N\} \neq \emptyset$, we must have $s = w_N$, but this contradicts (10). We have reached the final contradiction that proves (5) and therefore the lemma. \square

4 Conclusion

Theorem 1, that perfection and h -perfection are the same in plane triangulations, puts an end to further investigations of h -perfect plane triangulations: most of the interesting questions can be answered by appealing to their perfection.

In the larger class of planar graphs, however, perfection and h -perfection are no longer equivalent. Evidently, a 5-cycle is h -perfect but not perfect.

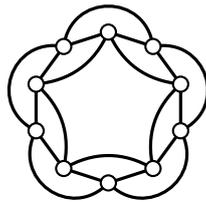


Figure 5: The graph C_{10}^2 .

Planarity is closed under taking t -contractions and induced subgraphs. Therefore, it might be possible to characterise those t -imperfect planar graphs that are minimal with respect to these two operations. So far, the known planar minimal obstructions are the odd wheels and C_{10}^2 (see Figure 5). Are these the only ones?

While general planar graphs obviously offer less structure than triangulations, some arguments might still be adapted. In particular, the key argument to derive Theorem 1 from Theorem 2 remains valid, and thus yields:

Proposition 7. *A planar graph is h -perfect if and only if each of its K_4 -free induced subgraphs is t -perfect.*

Therefore, as for plane triangulations, any characterisation of t -perfection in planar graphs in terms of forbidden induced subgraphs can be directly extended to h -perfection.

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Yohann Benchetrit <yohann.benchetrit@ulb.ac.be>
Université libre de Bruxelles, Belgium
Henning Bruhn <henning.bruhn@uni-ulm.de>
Universität Ulm, Germany