Pseudo-elliptic bundles, deformation data, and the reduction of Galois covers

Irene I. Bouw

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Introduction

The main objects we study in this paper are so called pseudo-elliptic bundles. These are certain filtered flat vector bundles $(\mathcal{E},\nabla)$ of rank 2 over a smooth projective curve B_0/k, where k is an algebraically closed field of characteristic $p>0$. These pseudo-elliptic bundles are a variation on elliptic crystals, as studied by Ogus [38]. We quickly recall how elliptic crystals arise naturally from families of elliptic curves. Let $f:E_0\to B_0$ be a non-isotrivial family of semistable curves. Then the de Rham cohomology $\mathcal{H}=\mathcal{H}^1_{\rm dR}(E_0/B_0)$ of E_0 defines a flat vector bundle $(\mathcal{H},\nabla)$, where

$$\nabla:\mathcal{H}\longrightarrow\mathcal{H}\otimes\Omega^{\rm log}_{B_0/k}$$

is the Gauß–Manin connection; in this situation it has at most regular singularities. The flat vector bundle $(\mathcal{H},\nabla)$ is an example of what we call an elliptic bundle. In Section 4.12 we review this situation and consider a concrete example, following [44].

Our pseudo-elliptic bundles are somewhat more general. Essentially, we keep all the structure that $\mathcal{H}$ has except the Serre duality. More precisely, a pseudo-elliptic bundle $\mathcal{E}$ is a vector bundle of rank two, endowed with a connection $\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega^{\rm log}_{B_0/k}$ and a nontrivial filtration ${\rm Fil}^1(\mathcal{E})\subset\mathcal{E}$. We require that the associated Kodaira–Spencer map is nonzero, and that the associated p-curvature is nilpotent and nonzero. We refer to Section 4.1 for a precise definition. One difference between our approach and that of Ogus is that our bundles $\mathcal{E}$ live in characteristic p, i.e. we restrict ourselves to reduction of the crystal modulo p.

We construct a class of pseudo-elliptic bundles which are associated to a deformation datum over $k(B_0)$. Deformation data arise naturally from the theory of stable reduction of Galois covers of curves, but may be studied independently. A deformation datum consists of a Galois cover $g_k:Z_k\to\mathbb{P}^1_k$ of degree prime to p, together with a differential form ω on Z_k satisfying certain conditions (Section 2.1). The pseudo-elliptic bundle $\mathcal{E}$ we associate to a deformation datum is a subbundle $\mathcal{E}\subset\mathcal{H}^1_{\rm dR}(Z_k/k(B_0))_\chi$ of an isotypical subspace of the de Rham cohomology. Here $\chi:H_0:={\rm Gal}(Z_k,\mathbb{P}^1_k)\to\mathbb{F}_p^\times$ is a character which comes with the deformation datum. In general, the rank of $\mathcal{H}^1_{\rm dR}(Z_k/k(B_0))_\chi$ will be larger than 2. With few exceptions, the Serre duality does not induce a natural duality on $\mathcal{E}$.

A deformation datum (g_k,ω) corresponds to a $\mathcal{G}$-torsor $Y_k\to\mathbb{P}^1_k$, where $\mathcal{G}$ is a finite flat group scheme which is generically isomorphic to $\boldsymbol{\mu}_p\rtimes_\chi H_0$. This construction allows a concrete description of the flat bundle $(\mathcal{E},\nabla)$ in terms of certain expansion coefficients. These expansion coefficients generalize the Hasse invariant for families of elliptic curves.

The kind of deformation data we are interested in here we call *special deformation data*; they are introduced in Section 2. In Section 3 we study the existence of special deformation data with fixed local invariants. We show that the existence of such a deformation datum is equivalent to the existence of a polynomial solution of fixed degree of a differential equation with fixed restricted Riemann scheme, where we need to impose a further condition on part of the singularities. We use this description to show the existence of special deformation data, for a certain class of local invariants. We also solve an analog of Dwork's accessary parameter problem. This relies on a technique due Wewers ([49], [51]) for lifting the $\mathcal{G}$-torsor $Y_k \to \mathbb{P}^1_k$ to characteristic zero. This is rather delicate, as the map $Y_k \to \mathbb{P}^1_k$ is inseparable.

Our main results concern the study of the properties of pseudo-elliptic bundles. For example, we give a criterion for the Kodaira–Spencer map to be an isomorphism. If this holds, our bundles are indigenous bundles in the sense of [10]. In the terminology of Mochizuki [36, Section I.4] these are flat vector bundles whose associated projective bundle is torally indigenous. Just as in the classical case of families of elliptic curves, the theorem on the Kodaira–Spencer map relies on the deformation theory. In our case this is the deformation theory of deformation data in characteristic p. As explained before, a deformation datum corresponds to a torsor under a finite flat group scheme. The study of the deformation problem relies on work of Wewers [51] on $\boldsymbol{\mu}_p$-torsors.

The last part of the paper (Sections 5 and 6) concern applications to the theory of stable reduction of Galois covers and the Hurwitz spaces parameterizing these. Let G be a finite group whose order is strictly divisible by p. Let R be a complete discrete valuation ring with fraction field K of characteristic zero and residue field an algebraically closed field k of characteristic $p > 0$. We start with a G-Galois cover $f : Y \to X = \mathbb{P}^1_K$ branched at four ordered points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$. We suppose that $(X; x_i)$ is generic; this means that λ is transcendental over $\mathbb{Q}_p$. One may define the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ of f (Section 2.2); $\bar{f}$ is a finite G-equivariant map of semistable curves over k which will in general be inseparable over certain irreducible components of $\bar{X}$. We say that f has *bad reduction* if there exists a component X_i of $\bar{X}$ such that the restriction of $\bar{f}$ to X_i is inseparable. Otherwise, we say that f has *good reduction*.

Suppose that f has bad reduction to characteristic $p > 0$. The conditions that p strictly divides the order of G and that $(X; x_i)$ is generic imply that the stable reduction $\bar{f}$ is well understood, by an extension of results of Raynaud [40] and Wewers [50]. All information is encoded in a deformation datum $(g_k : Z_k \to \mathbb{P}^1_k, \omega)$. If the deformation datum is special, we may associate to f a cover $B_0 \to \mathbb{P}^1_\lambda$ together with a flat vector bundle $(\mathcal{E}, \nabla)$ on B_0. An additional condition guarantees that $\mathcal{E}$ is pseudo-elliptic.

Let $\mathcal{H}_G/\mathbb{Q}_p$ be the (inner) Hurwitz space parameterizing G-Galois covers of $\mathbb{P}^1$ branched at four ordered points. Let $\mathcal{H}$ be the connected component of $\mathcal{H}_G$ such that f corresponds to a point of $\mathcal{H}$. We write $\pi : \mathcal{H} \to \mathbb{P}^1_\lambda$ for the natural map which sends a G-Galois cover branched at $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$ to λ. The relation between the flat bundle $(\mathcal{E}, \nabla)$ and the stable reduction of ϖ is explained in Section 5.1. We interprete $(\mathcal{E}, \nabla)$ as a Swan conductor in the sense of Kato [22] associated to the stable model of ϖ. The theory of the Swan conductor provides a replacement for the structure of deformation data in the case that the Sylow p-subgroup of the Galois group is larger than p. However, this theory has not been worked out yet. This paper gives a first glimpse on what the theory might look like.

The idea that there is a relation between the stable reduction of the Hurwitz space $\mathcal{H}$ and the stable reduction of the G-Galois covers f parameterized by $\mathcal{H}$ has also been used in [8]. In that paper G is isomorphic to $\mathbb{Z}/p \rtimes \mathbb{Z}/m$. The cover $\varpi : \mathbb{H} \to \mathbb{P}^1_K$ is branched at three points and p divides the order of the Galois group Γ of ϖ at most once. This allows to apply the techniques of Raynaud and Wewers both to the cover ϖ and the G-Galois covers f.

We now give more precise description of our results.

Deformation data Let $f : Y \to X = \mathbb{P}^1_K$ be a G-Galois cover over K branched at four points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$, such that $(\mathbb{P}^1_K; x_i)$ is generic. Assume that f has bad reduction to characteristic p. For simplicity, we assume that the ramification indices of f are prime to p. To the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ of f we may associate a *deformation datum* (g_k, ω). Here $g_k : Z_k \to \mathbb{P}^1_k$ is a Galois cover of $\mathbb{P}^1_k$ of order prime to p, and ω is a differential form on Z_k which is logarithmic, i.e. $\omega = \mathrm{d}g/g$ (Proposition 2.3.3). The curve $\mathbb{P}^1_k$ is the reduction of X. The points (x_i) specialize to pairwise distinct points (τ_i) on $\mathbb{P}^1_k$. The cover g_k is branched at $\tau_0, \ldots, \tau_3$, together with additional points $(\tau_i)_{i\in\mathbb{B}_{\rm new}}$. It can be shown that since $(X; x_i)$ is generic the deformation datum determines the stable reduction. There is a character $\chi : H_0 := \mathrm{Gal}(Z_k, \mathbb{P}^1_k) \to \mathbb{F}_p^\times$ such that $h \cdot \omega = \chi(h)\omega$. To simplify the exposition in this introduction, we suppose that χ is injective, i.e. H_0 is a cyclic group of order dividing $p-1$. We curve Z_k is a connected component of the smooth projective curve defined by the Kummer equation

$$z^{p-1} = \prod_{\tau_i\neq\infty} (x - \tau_i)^{a_i},$$

where $0 < a_i < p-1$. We call $\boldsymbol{\sigma} = (\sigma_i := a_i/(p-1))$ the *signature* of the deformation datum.

Let $\kappa = \bar{\kappa}$ be an extension of k over which the deformation datum (g_k, ω) may be defined. Denote by $H^1_{\rm dR}(Z_k/\kappa)$ the first de Rham cohomology group in characteristic p. The group H_0 acts on it; write $H^1_{\rm dR}(Z_k/\kappa)_\chi$ for the subspace on which H_0 acts via the character χ. Consider its Hodge filtration:

$$0 \longrightarrow \mathrm{Fil}^1 = H^0(Z_k, \Omega)_\chi \longrightarrow H^1_{\rm dR}(Z_k/\kappa)_\chi \longrightarrow H^1(Z_k, \mathcal{O})_\chi \longrightarrow 0.$$

An easy computation shows that $\dim H^1(Z_k, \mathcal{O})_\chi \leq 1$. (This uses the assumption that f is branched at four points, see Section 2.3.) Generalizing the terminology of [49], we call a deformation datum *special* if $\dim H^1(Z_k, \mathcal{O})_\chi = 1$. In terms of the signature, this condition correspond to $\sum_i a_i = 2(p-1)$.

Sections 2 and 3 concern the study of deformation data. We are interested in the existence of special deformation data with giving signature, and properties of the deformation space of deformation data. We describe the main results. In Section 3.1 and Section 3.2 we show that there is a bijection between special deformation data of given signature $\boldsymbol{\sigma}$ and polynomial solutions $u = u(x)$ of degree $d = (p-1) - (\sum_i a_i)/2$ of a certain Fuchsian differential equation (23) satisfying certain additional properties (Proposition 3.2.3). The set of singularities of this differential equation is the set (τ_i) of branch points of g_k. The differential equation depends furthermore on a set (β_j) of *accessary parameters*. The following result is proved in Section 3.3.

Proposition 3.3.2: Suppose that $a_i = 2$ for all $i \notin \{0, 1, 2, 3\}$. Then there exists a deformation datum with signature $\boldsymbol{\sigma}$.

The situation of the proposition is the easiest case; here there is just one accessary parameter. For more general signatures the answer to the existence question is more subtile.

Our next result is an analog to Dwork's accessary parameter problem (Section 3.4). Fix a signature $\boldsymbol{\sigma}$ as above, and suppose that there exists a special deformation datum with signature $\boldsymbol{\sigma}$. We define a variety B_0 essentially as the locus of all $(\tau_i, \beta_j)_{i,j}$ such that there exists a special deformation datum with the fixed signature $\boldsymbol{\sigma}$, branch locus (τ_i) and accessary parameters (β_j). NOT STATED AS SUCH!!!

Theorem The natural map $\pi : B_0 \to \mathbb{P}^1_\lambda$ is finite and separable.

This result is proven in Section 3.4. It relies on studying the deformation theory of $\boldsymbol{\mu}_p$-torsors ([51]).

The pseudo-elliptic bundle associated to a deformation datum Let $(g_k : Z_k \to \mathbb{P}^1_k, \omega)$ be a deformation datum as above, where $(\mathbb{P}^1_k; \tau_0 = \infty, \tau_1 = 0, \tau_2 = 1, \tau_3 = \lambda)$ is generic. The following is our key assumption in most of this paper:

Assumption 4.2.1:

- $\dim H^1(Z_k, \mathcal{O})_\chi = 1$, i.e. the deformation datum is special,

- the Frobenius morphism $F : H^1(Z_k, \mathcal{O})_\chi \to H^1(Z_k, \mathcal{O})_\chi$ is a isomorphism.

This assumption allows us to define a filtered flat vector bundle $(\mathcal{E}, \nabla)$ on the (smooth projective) curve B_0. The differential form ω is a holomorphic logarithmic differential form on Z_k, therefore it lies in $\mathrm{Fil}^1 \subset H^1_{\mathrm{dR}}(Z_k/k(B_0))_\chi$. In Section 4.3 we define a 2-dimensional vector subspace $\bar{V} := \mathcal{E} \otimes k(B_0) \subset H^1_{\mathrm{dR}}(Z_k/k(B_0))_\chi$ which is generated by ω and a suitable lift of $H^1(Z_k, \mathcal{O})_\chi$. We show that $\bar{V} := \mathcal{E} \otimes k(B_0)$ is stabilized by the Gauss-Manin connection $\nabla : H^1_{\mathrm{dR}}(Z_k/k(B_0))_\chi \to H^1_{\mathrm{dR}}(Z_k/k(B_0))_\chi \otimes \Omega^{\log}_{B_0/k}$. This gives $\bar{V}$ the structure of (the reduction modulo p of) an F-crystal. The Hodge filtration on $H^1_{\mathrm{dR}}(Z_k/k(B_0))_\chi$ induces a nontrivial filtration on V.

The reason for imposing Assumption 4.2.1 is the following. If we drop the assumption that the deformation datum is special, the dimension of $H^1(Z_k, \mathcal{O})_\chi$ is zero, and the analog of the bundle $\mathcal{E}$ has rank one. If the deformation datum is special but $F : H^1(Z_k, \mathcal{O})_\chi \to H^1(Z_k, \mathcal{O})_\chi$ is identically zero, one may define an analog of the bundle $\mathcal{E}$ as well. (The definition we give in Section 4.3 does not go through, but one may use the correspondence between $\mathcal{E}$ and the group scheme $\mathcal{G}$ defined in Section 4.4.) In this case the bundle $\mathcal{E}$ has rank 2, but it splits as a direct sum $\mathcal{E} \simeq \mathrm{Fil}^1 \oplus \mathcal{M}$ of flat vector bundles. This is in some sense a degenerate case; one expects that it only occurs rarely, if at all. To get an interesting theory, it is certainly not enough to study the flat vector bundle $(\mathcal{E}, \nabla)$ which lives in characteristic p. Probably, one would be able to extend some of the results by replacing the mod p F-crystal $\bar{V} = \mathcal{E} \otimes k(B_0)$, by the full F-crystal V, if it exists.

Sections 4.4–4.10 are the heart of the paper. They concern the properties of the flat vector bundle $(\mathcal{E}, \nabla)$. The following theorem summarizes the main results.

Theorem

(a) The vector space $\bar{V}$ extends to a pseudo-elliptic bundle $(\mathcal{E}, \nabla)$.

(b) Under a mild hypotheses, the associated Kodaira–Spencer map is an isomorphism, and $\mathcal{E}$ is an indigenous bundle.

The study of the Kodaira–Spencer map again relies on a study of the deformation theory of $\boldsymbol{\mu}_p$-torsors. Since the corresponding theory for $\boldsymbol{\alpha}_p$-torsors is not available, we need to impose a mild condition here.

It is well know that the flat vector bundle $(\mathcal{E}, \nabla)$ corresponds to a Fuchsian differential equation. We explicitly calculate this differential equation in terms of certain expansion coefficients of a basis of $\bar{V}$, following Katz ([27], [28]). Let us explain the idea. The differential form ω is not defined over $k(B_0)$, but we may write $\omega = \Phi_*^{1/(p-1)} \omega_0$ with $\Phi_* \in k(B_0)$ and $\omega_0 = z\,\mathrm{d}x/x(x-1)(x-\lambda)$, i.e. ω_0 may be defined over $k(B_0)$. The rational function Φ_* may be interpreted as an expansion coefficient of ω_0. It is an analog to our situation of the Hasse invariant in the case that $g_k : Z_k \to \mathbb{P}^1_k$ is the Legendre family of elliptic curves. We also define a "dual" function Φ; it corresponds in a similar way to a suitably chosen basis of $H^0(Z_k, \Omega^1)_{\chi^{-1}} = H^1(Z_k, \mathcal{O})_\chi^{\mathrm{dual}}$.

In Section 5 we change focus. We let $f : Y \to \mathbb{P}^1$ be a G-Galois cover with bad reduction, whose stable reduction corresponds to the deformation datum (g_k, ω). As before, we let $\mathcal{H} = \mathcal{H}_f$ be the connected component of the Hurwitz space $\mathcal{H}_G$ of G-Galois covers such that f corresponds to a point of $\mathcal{H}$. We show that the cover $\pi : \mathcal{H} \to \mathbb{P}^1_\lambda$ has bad reduction.

Denote by $\varpi : \mathbb{H} \to \mathbb{P}^1_\lambda$ the Galois closure of π. Since π has bad reduction, it follows that p divides the order of the Galois group Γ of ϖ. In general, the order of the Sylow p-subgroup of Γ is larger than p. This implies that the stable reduction of $\bar{\varpi}$ is no longer described by a deformation datum. We use Kato's theory of differential Swan conductors as a replacement (Section 5.1). Essentially, in stead of one deformation datum one gets a set of Swan conductors.

Theorem 5.3.2 The bundle $(\mathcal{E}, \nabla)$ describes a Swan conductor of the cover $\varpi : \mathbb{H} \to \mathbb{P}^1_\lambda$.

In case p strictly divides the order of Γ, there is essentially only one Swan conductor associated to ϖ, which is just the deformation datum, in the sense we considered it before. It follows therefore from the results of [50] that this Swan conductor completely determines the stable reduction of ϖ. (This holds in the case studied in [8].) It is not known how much information the bundle $\mathcal{E}$ gives on the stable reduction of ϖ. In case P is elementary abelian one might hope to be able to say something. For this one needs to work out the theory of Swan conductors in the setting of stable reduction, generalizing the results of [40] and [50].

Section 6 contains complements and examples. We give a sufficient conditions on the H-Galois cover $f : Y \to \mathbb{P}^1$ for the conditions we imposed to hold. In case G is $\mathrm{SL}_2(p)$ or $\mathrm{PSL}_2(p)$ we give a concrete example. This illustrates how one can use our results to compute the number of G-Galois covers with good reduction, generalizing results of [12].

Section 1 is a bit independent of the rest of the paper. It considers the definition of the F-crystal V in case $G = \mathbb{F}_q \rtimes_\chi \mathbb{Z}/m$, where $\chi : \mathbb{Z}/m \to \mathbb{F}_q^\times$ is an irreducible character. A major difference here is that we do not suppose that p strictly divides the order of G. We focus on defining the analog of the expansion coefficient Φ_*, Φ, and studying there properties. We do not consider the relation with the stable reduction of the Hurwitz space of G-Galois covers here. This will be done elsewhere, generalizing the result of [8] in the case $q = p$. This section also recalls some p-adic limit formulas for the eigenvalues of the Frobenius morphism on the crystal V. This is a mixed characteristic analog of the expansion coefficient Φ we discussed above. It seems that similar formulas exist in the more general context of pseudo-elliptic bundles, but for this one needs the mod p description of this paper to mixed characteristics. The work of Katz [28] and Ogus [38] suggest that this may be done.

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1 The Picard–Fuchs differential equation of a cyclic cover of $\mathbb{P}^1$

This chapter serves as an introduction to the rest of the paper. We recall known results on cyclic covers of the projective line in mixed characteristic. In Section 1.1 we describe the de Rham cohomology of a cyclic cover of $\mathbb{P}^1$ in characteristic zero, and in Section 1.3 we recall the definition of the unit root F-crystal. Starting from Section 1.2, we suppose that $g : Z \to \mathbb{P}^1$ is an m-cyclic cover branched at four points in mixed characteristic p, where m is prime to p. We then study the eigenvalues of the Frobenius morphism on the de Rham cohomology of Z, and relate them to solutions of the Picard–Fuchs differential equation of Z. These results are all known, but we present them in a coherent way, which will motivate the rest of the paper. An interesting aspect which will not addressed in the rest of the paper is the p-adic limit formulas for the eigenvalues of Frobenius (see for example Proposition 1.3.6). Similar formulas exist for the more general situation we consider later on. This illustrates that the picture of Section 4 is the first approximation of an F-crystal which carries a rich arithmetic structure.

1.1 The de Rham cohomology of a cyclic cover of $\mathbb{P}^1$ Let $m > 1$ and $r \geq 3$ be integers. Let $S = \mathrm{Spec}(A)$ be smooth over $\mathrm{Spec}(\mathbb{Z}[\zeta_m, 1/m])$, where ζ_m is a primitive mth root of unity. Write K for the fraction field of A. Choose an injective character $\chi : \mathbb{Z}/m \to K$. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be an r-tuple of integers with $0 < a_i < m$ and $\sum a_i \equiv 0 \bmod m$. We suppose that $\gcd(a_1, \ldots, a_r, m) = 1$. Let $\mathbf{x} = (x_1 = 0, x_2 = 1, x_3, \ldots, x_r = \infty)$ be pairwise disjoint S-valued points of $\mathbb{P}^1_S$. Let $Z \to \mathbb{P}^1_S$

be the m-cyclic cover of type $(\mathbf{a}; x)$ ([7, Definition 2.1]). This means that Z is the complete nonsingular curve over S corresponding to the equation

$$z^m = \prod_{\mu=1}^{r-1} (x - x_\mu)^{a_\mu},$$

where $h \in H := \mathrm{Gal}(Z, \mathbb{P}^1) \simeq \mathbb{Z}/m$ acts as $h \cdot z = \chi(h) z$.

Notation 1.1.1 Write $\sum_{\mu=1}^{r} a_\mu = (b+1)m$ and $\sigma_\mu = a_\mu/m$. For a rational number ν, we denote by $\langle \nu \rangle$ its fractional part and by $[\nu]$ its integral part. For $i = 1, \ldots, m-1$, we write $a_\mu(i) = m\langle i \cdot a_\mu/m \rangle$. Let $\sum_{\mu=1}^{r} a_\mu(i) = (b(i)+1)m$ and $\sigma_\mu(i) = a_\mu(i)/m$.

The type depends on the choice of the character χ. Replacing χ by χ^i with $\gcd(i, m) = 1$ changes $(a_1, \ldots, a_r)$ into $(a_1(i), \ldots, a_r(i))$.

Denote by $H^1_{\mathrm{dR}}(Z/S)$ for the first relative de Rham cohomology group of Z/S and write $H^1_{\mathrm{dR}}(Z/S)_{\chi^i}$ for the eigenspace corresponding to χ^i. Recall that the Hodge filtration looks in this case as follows:

$$0 \to H^0(Z, \Omega^1)_\chi \to H^1_{\mathrm{dR}}(Z/S)_\chi \to H^1(Z, \mathcal{O}_Z)_\chi \to 0. \tag{1}$$

The following lemma describes a basis of $H^1_{\mathrm{dR}}(Z/S)_\chi$.

Lemma 1.1.2 *Let $0 < i < m$ be an integer prime to m.*

(a) The dimension of $H^1_{\mathrm{dR}}(Z/S)_{\chi^i}$ (resp. $H^0(Z, \Omega^1)_{\chi^i}$) is $r-2$ (resp. $r-2-b(i)$).

(b) The differentials

$$\omega_j^i = \frac{x^{j-1} z^i \mathrm{d}x}{\prod_{\mu=1}^{r-1} (x - x_\mu)^{1+[i\,\sigma_\mu]}}, \qquad j = 1, \ldots, r_i - 2$$

form a basis over S of $H^1_{\mathrm{dR}}(Z/S)_{\chi^i}$.

(c) The differentials $\omega_1, \ldots, \omega_{r-2-b(i)}$ form a basis over S of $H^0(Z, \Omega^1)_{\chi^i}$.

Proof: It is shown in [7, Lemma 4.3] that the dimension over S of $H^1(Z, \mathcal{O})_{\chi^i}$ is $b(i)$. Serre duality implies that $H^1(Z, \mathcal{O})_{\chi^i}$ is dual to $H^0(Z, \Omega^1)_{\chi^{-i}}$. Therefore $H^0(Z, \Omega^1)_{\chi^{-i}}$ has S-dimension $(a_1(m-i) + \cdots a_r(m-i) - m)/m = r - 2 - b(i)$. This proves (a).

Write $Z_{\bar{K}}$ (resp. $S_{\bar{K}}$) for the geometric generic fiber of Z (resp. S). Part (b) follows for example by considering $H^1_{\mathrm{dR}}(Z_{\bar{K}}/S_{\bar{K}})_{\chi^i}$. Since $\bar{K}$ has characteristic zero, this space consists of differentials of the second kind modulo exact differentials, [16, Section 5.3]. (Recall that a meromorphic differential on $Z_{\bar{K}}$ is *of the second kind* if it does not have any residues.) Part (c) follows from (a) and (b). □

The following lemma describes a basis of $H^1(Z, \mathcal{O})_\chi$ using Čech cohomology with respect to the covering $\mathcal{U} = \{U_1, U_2\}$ of Z, where $U_1 = Z - \{0\}$ and $U_2 = z - \{\infty\}$.

Lemma 1.1.3 *Let $0 < i < m$ be an integer prime to m.*

(a) For $j = 1, \ldots, b_i$, let

$$\xi_j^i = \frac{z^i}{x^j \prod_{\mu=1}^{r-1} (x - x_\mu)^{[i\,\sigma_\mu]}}.$$

Then $(\xi_j^i)_j$ is a basis of $H^1(Z, \mathcal{O})_{\chi^i}$. This is the dual basis to (ω_j^{m-i}) with respect to Serre duality.

(b) Let ω be a differential of the second kind which is holomorphic outside ∞. There exists a unique rational function f on Z such that $\omega + \mathrm{d}f$ is holomorphic at $Z - \{0\}$. The map

$$H^1_{\mathrm{dR}}(Z/S) \to H^1(Z, \mathcal{O})$$

of (1) sends the class of ω to the class of f.

Proof: The fact that $(\xi^i_j)_j$ form a basis of $H^1(Z,\mathcal{O})_{\chi^i}$ is shown in [7, Section 5]. We check that this basis is dual to $(\omega^{m-i}_j)_{j=1,\dots,r-2-b_i}$ under Serre duality. Recall from [42, Section 8] that Serre duality is given by the pairing

$$H^0(Z,\Omega^1)_{\chi^i} \times H^1(Z,\mathcal{O})_{\chi^{m-i}}, \qquad \langle\, \omega, \xi\,\rangle = \sum_{P\in Z} \mathrm{Res}_P(\xi\omega).$$

One checks that $\langle \omega^i_j, \xi^{m-i}_{j'}\rangle = \delta_{j,j'}$. This proves (a).

Let ω be a differential of the second kind on $Z_{\bar{K}}$. Suppose that the class of ω represents an element of $H^1_{\mathrm{dR}}(Z/S)_{\chi^i}$. Lemma 1.1.2 implies that the class of ω is represented by an differential of the second kind which is holomorphic outside ∞. The existence of a rational function f as in the statement of (b) follows from now [16, page 456]. Since $[\omega] \in H^1_{\mathrm{dR}}(Z_{\bar{K}}/S_{\bar{K}})_{\chi^i}$, the rational function f may be written as $z^i f_1/f_2$, where the f_μ are polynomials in x. It is easy to see that the map described in the statement of the lemma is surjective and has kernel $H^0(Z_{\bar{K}}, \Omega^1)$. Part (b) now follows from the fact that Poincaré duality is compatible with Serre duality. □

In the rest of this section we suppose that $r = 4$ and denote the branch point of the cover by $x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty$. We let $S = \mathrm{Spec}(\mathbb{Z}[\zeta_m, \lambda, 1/m\lambda(\lambda-1)])$. We compute the action of the Gauß–Manin connection $\nabla : H^1_{\mathrm{dR}}(Z/S)_{\chi^i} \to H^1_{\mathrm{dR}}(Z/S)_{\chi^i} \otimes_S \Omega^1_S$. The results of this section are probably known to the experts. Some of it goes back to Dwork [15] and Stienstra–Van der Put–Van der Marel [45]. The case $m = 2$ is well known, and can be found for example in [23].

We restrict to the case $r = 4$, since then the base space S is one dimensional and the Picard–Fuchs differential equation is an ordinary differential equation of order two. The computation of the Picard–Fuchs differential equation can be easily extended to the general case. Let $0 < i < m$ be an integer which is prime to m. Write

$$\omega^i_1 \;=\; \frac{z^i \mathrm{d}x}{x^{1+[i\sigma_1]}(x-1)^{1+[i\sigma_2]}(x-\lambda)^{1+[i\sigma_3]}} = \frac{\mathrm{d}x}{x^{1-\sigma_1(i)}(x-1)^{1-\sigma_2(i)}(x-\lambda)^{1-\sigma_3(i)}}, \tag{2}$$

$$(\omega^i_1)' \;=\; \nabla(\frac{\mathrm{d}}{\mathrm{d}\lambda})\,\omega^i_1 = (1-\sigma_3(i))\frac{\omega^i_1}{(x-\lambda)}, \qquad (\omega^i_1)'' := \nabla(\frac{\mathrm{d}}{\mathrm{d}\lambda})\,(\omega^i_1)'. \tag{3}$$

Recall from Section 1.1 that $H^1_{\mathrm{dR}}(Z/S)_{\chi^i}$ has S-dimension two. The S-dimension of $H^0(Z,\Omega^1)_{\chi^i}$ is $2 - b(i) = 3 - (a_1(i) + a_2(i) + a_3(i) + a_4(i))/m \le 2$. In other words, the Hodge filtration is "trivial" unless $b(i) = 1$. One checks that $\omega(i)$ (resp. $\omega(i)'$) is holomorphic if and only if $b(i) \le 1$ (resp. $b(i) = 0$).

Lemma 1.1.4 *Let $0 < i < m$ be an integer prime to p. Write $A^*(i) = 1 - \sigma_3(i)$, $B^*(i) = 2 - (\sigma_1(i) + \sigma_2(i) + \sigma_3(i))$, and $C^*(i) = 2 - (\sigma_1(i) + \sigma_3(i))$. Put $\omega := \omega^i_1$. Then*

$$\lambda(\lambda-1)\omega'' + [(A^*(i) + B^*(i) + 1)\lambda - C^*(i)]\,\omega' + A^*(i)B^*(i)\omega_1 = 0 \in H^1_{\mathrm{dR}}(Z/S)_{\chi^i}. \tag{4}$$

Proof: It follows from Lemma 1.1.2 that ω and ω' form a basis of $H^1_{\mathrm{dR}}(Z/S)_{\chi^i}$. The lemma now follows from the identity

$$\lambda(\lambda-1)\omega'' + [(A^*(i) + B^*(i) + 1)\lambda - C^*(i)]\,\omega' + A^*(i)B^*(i)\omega = (\sigma_3(i)-1)\mathrm{d}\frac{x^{\sigma_1(i)}(x-1)^{\sigma_2(i)}}{(x-\lambda)^{2-\sigma_3(i)}}.$$

□

We let $\tilde{\chi} = \chi + \chi^p + \cdots + \chi^{p^{f-1}}$. Then

$$H^1_{\mathrm{dR}}(Z/S)_{\tilde{\chi}} := \oplus_{i=0}^{f-1} H^1_{\mathrm{dR}}(Z/S)_{\chi^{p^i}}$$

is an F-crystal. This is seen as follows. Denote by F the absolute Frobenius morphism. By the comparison isomorphism between $H^1_{\mathrm{cris}}(Z/S)$ and $H^1_{\mathrm{dR}}(Z/S)$, we obtain a p-semilinear map $F : H^1_{\mathrm{dR}}(Z/S) \to H^1_{\mathrm{dR}}(Z/S)$. Restricting to the eigenspaces, yields a map $F : H^1_{\mathrm{dR}}(Z/S)_{\chi^i} \to H^1_{\mathrm{dR}}(Z/S)_{\chi^{pi}}$. Therefore we obtain a map $F : H^1_{\mathrm{dR}}(Z/S)_{\tilde{\chi}} \to H^1_{\mathrm{dR}}(Z/S)_{\tilde{\chi}}$ and $H^1_{\mathrm{dR}}(Z/S)_{\tilde{\chi}}$.

1.2 The Frobenius morphism in characteristic p Let p be prime which is prime to m. Denote by f the order of p in $\mathbb{Z}/m^*$ and write $q = p^f$. In this section we compute the action of the Frobenius morphism on de Rham cohomology in characteristic p. Essentially, this follows from results of [7]. In this section, we use the notation from Section 1.1 and suppose that $r = 4$. Write $\bar{Z} = Z \otimes \mathbb{F}_p$. For $t \in \mathbb{P}^1 - \{0, 1, \infty\}$, we write $\bar{Z}_t$ for the fiber at t of $\bar{Z}$.

Lemma 1.2.1 *Suppose that $b(p^i) = 1$ for $i = 0, \ldots, f-1$. There exists a dense open subset $\mathcal{U} \subset \mathbb{P}^1 - \{0, 1, \infty\}$, such that map $F : H^1(\bar{Z}_t, \mathcal{O})_{\tilde{\chi}} \to H^1(\bar{Z}_t, \mathcal{O})_{\tilde{\chi}}$ is an isomorphism for all $t \in \mathcal{U}$.*

Proof: This follows from [7, Proposition 6.7]. □

Suppose that $b(p^i) = \dim H^1(\bar{Z}, \mathcal{O})_{\chi^{p^i}} = 1$, for $i = 0, \ldots, f-1$. It follows from Lemma 1.1.3 that $\xi(i) := \xi_1^{p^i}$ is a basis of $H^1(\bar{Z}, \mathcal{O})_{\chi^{p^i}}$. We write $\omega(i) := \omega_1^i$ for the basis of $H^0(\bar{Z}, \Omega^1)_{\chi^{p^i}}$.

Definition 1.2.2 Let $0 \leq i \leq f-1$ be an integer. The polynomial

$$\Phi_i(\lambda) = (-1)^{N_i} \sum_{n_1+n_2=N_i} \binom{[p\sigma_2(p^{i-1})]}{n_1} \binom{[p\sigma_3(p^{i-1})]}{n_2} \lambda^{n_2} \tag{5}$$

is called the *ith partial* **a***-Hasse invariant*, or ith Hasse invariant for short. Here $N_i = p - 1 - [p\sigma_4(p^{i-1})]$. The *Hasse invariant* is defined as $\Phi = \prod_{i=0}^{f-1} \Phi_i$.

It is shown in [7, Section 5] that $F\xi_{i-1} = \Phi_i \xi_i$. If $m = 2$, then the only possible type is $\mathbf{a} = (1, 1, 1, 1)$, and Z is the Legendre family of elliptic curves. In this case Φ_1 is the classical Hasse invariant, whose zeros are the supersingular λ's, i.e. the values of λ for which the elliptic curve $\bar{Z}_\lambda$ is supersingular.

The Cartier operator $\mathcal{C} : H^0(\bar{Z}, \Omega^1) \to H^0(\bar{Z}, \Omega^1)$ is defined as the transpose of $F : H^1(\bar{Z}, \mathcal{O}) \to H^1(\bar{Z}, \mathcal{O})$. This implies that $\mathcal{C}\omega(i) = (\Phi_i^*)^{1/p}\omega(i-1)$. Here Φ_i^* is the ith Hasse invariant corresponding to the *dual type* $\mathbf{a}^* = (m - a_1, m - a_2, m - a_3, m - a_4)$ or, equivalently, the matrix of $F : H^1(\bar{Z}, \mathcal{O})_{\chi^{-p^{i-1}}} \to H^1(\bar{Z}, \mathcal{O})_{\chi^{-p^i}}$.

One easily checks that Φ_i is a nonzero polynomial (Lemma 1.2.3.(d)). This remark proves Lemma 1.2.1.(b). The open set $\mathcal{U}$ mentioned in the statement of Lemma 1.2.1 consists of the complement in $\mathbb{P}^1 - \{0, 1, \infty\}$ of the zero locus of the polynomials Φ_i. Assume that $b(p^i) = 1$ for $i = 0, \ldots, f-1$. Then we can describe the group scheme $J(\bar{Z}_t)[p]_\chi$, for $t \in \mathcal{U}$. Recall that there exist integers $\epsilon(i), \nu(i)$ and a local-local group scheme $L(i)$ such that

$$J(\bar{Z}_t)[p]_{\chi^i} \simeq (\mathbb{Z}/p)^{\epsilon(i)} \times (\mu_p)^{\nu(i)} \times L(i).$$

Since the Frobenius morphism $F : H^1(\bar{Z}_t, \mathcal{O})_{\chi^{p^i}} \to H^1(\bar{Z}_t, \mathcal{O})_{\chi^{p^{i+1}}}$ is an isomorphism, we have that $\nu(p^i) = \dim_{\mathbb{F}_q} H^1(\bar{Z}_t, \mathcal{O})^{F^f}_{\chi^{p^i}} = 1$. The Cartier dual of $J(\bar{Z}_t)[p]_{\chi^{p^i}}$ is $J(\bar{Z}_t)[p]_{\chi^{-p^i}}$. One easily checks that $b(-p^i) = 1$ for $i = 0, \ldots, f-1$, also. Therefore $\epsilon(p^i) = \nu(-p^i) = 1$ and $L(p^i) = (0)$. Alternatively, one could use that $\epsilon(i) = \dim_{\mathbb{F}_q} H^0(\bar{Z}_t, \Omega^1)^{\mathcal{C}^f}_{\chi^{p^i}}$. Just as in the case $m = 2$, the zeros of the Hasse invariants corresponds to curves for which the group scheme $J(\bar{Z}_t)[p]_{\tilde{\chi}}$ contains a local-local piece.

The Hasse invariant Φ_i^* is an expansion coefficient of the differential $\omega(i)$. To ease notation, we only explain this for $i=0$ and m prime, but it is clear how to extend the formula's. Our argument is adopted from [27]. Choose $u = x^{-1/m}$; this is a local parameter of $\bar{Z}$ at ∞. Write

$$\begin{aligned}\omega(0) &= \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)} = \frac{\mathrm{d}x}{x^{1-\sigma_1}(x-1)^{1-\sigma_2}(x-\lambda)^{1-\sigma_3}}\\ &= -mu^{a_4}(1-u^m)^{\sigma_2-1}(1-\lambda u^m)^{\sigma_3-1}\frac{\mathrm{d}u}{u}\\ &= -m\sum_{n\geq 0} P_{mn+a_4}(\lambda)u^{nm+1}\frac{\mathrm{d}u}{u},\end{aligned}$$

for the expansion of ω with respect to u. Here we use that $\sigma_1+\sigma_2+\sigma_3+\sigma_4=2$. Note

$$P_{nm+a_4} = (-1)^n \sum_{i+j=n} \binom{\sigma_2-1}{i}\binom{\sigma_3-1}{j}\lambda^j. \tag{6}$$

In particular for $n = \left[p\sigma_4(p^{f-1})\right]$ we find

$$P_{nm+a_4}(\lambda) \equiv \Phi_1^* \pmod p.$$

The main result of [27] implies that Φ_i^* is a solution modulo p of the Picard–Fuchs differential equation of Lemma 1.1.4.(b). This is also easy to check directly from the explicit formula (5) for $\Phi_{\mathbf{a}}$.

For future reference, we state the following elementary lemma.

Lemma 1.2.3 *(a) The polynomial Φ_i is a solution modulo p of the differential equation*

$$\lambda(\lambda-1)u'' + [(A^*(m-1)+B^*(m-1)+1)\lambda - C^*(m-1)]u' + A^*(m-1)B^*(m-1)u = 0,$$

where $A^*(i), B^*(i)$ and $C^*(i)$ are as in Lemma 1.1.4.

(b) At $\lambda = 0$, the polynomial Φ_i has a zero of order $\max(N_i - [p\sigma_2(p^{i-1})], 0)$.

(c) At $\lambda = 1$, the polynomial Φ_i has a zero of order $\max(N_i - [p\sigma_1(p^{i-1})], 0)$.

(d) The degree of Φ_i is $\min(N_i, [p\sigma_3(p^{i-1})])$.

(e) The polynomial Φ_i is nonzero as polynomial in λ. All zeros different from $\lambda = 0, 1$ are simple zeros.

(f) We have

$$\frac{\Phi_i'}{\Phi_i} \equiv \frac{(\Phi_i^*)'}{\Phi_i^*} - \frac{C^*(i)-1}{\lambda} + \frac{C^*(i)-A^*(i)-B^*(i)}{\lambda-1} \pmod p.$$

Proof: We already proved (a). Parts (b), (c) and (d) follow from the definition of Φ_i as in [8, Corollary 5.5]. Since $0 \leq [p\sigma_\mu(p^{i-1})] \leq p-1$ for all μ, the polynomial Φ_i is nonzero. All zeros of Φ_i different from 0 and 1 are simple, since Φ_i is the solution to a hypergeometric differential equation. The Cartier dual of the group scheme $J(\bar{Z}_\lambda)[p]_{\chi^{p^i}}$ is $J(\bar{Z}_\lambda)[p]_{\chi^{p^{-i}}}$. This implies that the zeros of Φ_i and Φ_i^* different from 0 and 1 are equal. Therefore (f) follows from (b) and (c). □

Definition 1.2.4 Let $A, B, C \in \mathbb{Q}$ and consider the corresponding hypergeometric differential equation

$$\lambda(\lambda-1)f'' + [(A+B+1)\lambda - C]f' + ABf = 0. \tag{7}$$

Define the *hypergeometric differential equation dual to (7)* as

$$\lambda(\lambda-1)f'' + [(A+B-3)\lambda + C - 2]f' + (A-1)(B-1)f = 0. \tag{8}$$

One easily checks that the dual to the Picard–Fuchs differential equation of $H^1_{\mathrm{dR}}(Z/S)_\chi$ is the Picard–Fuchs differential equation of $H^1_{\mathrm{dR}}(Z/S)_{\chi^{-1}}$. Both differential equations play a role in the description of the unit root crystal in Section 1.3. We write $A(i) = \sigma_3(i) = 1 - A^*(i)$, $B(i) = \sigma_1^i + \sigma_2^i + \sigma_3^i - 1 = 1 - B^*(i)$, $C(i) = \sigma_1^i + \sigma_3^i = 2 - C^*(i)$ for the parameters of the differential equation

$$\lambda(\lambda-1)\omega(i)'' + [(A(i)+B(i)+1)\lambda - C(i)]\omega(i)' + A(i)B(i)\omega(i) = 0 \tag{9}$$

dual to (4).

1.3 The unit root crystal The notations are as in Section 1.2. In particular, we suppose that $r = 4$ and $b(p^i) = 1$ for $i = 0, \dots, f-1$. Choose a lift $\tilde{\Phi}_i \in \mathbb{Z}[\lambda]$ of Φ_i. Put $\tilde{\Phi} = \prod_{i=0}^{f-1} \tilde{\Phi}_i$. Define R (resp. R_{ord}) to be the p-adic completion of $\mathbb{Z}_p[\lambda][1/\lambda(\lambda-1)]$ (resp. $\mathbb{Z}_p[\lambda][1/\lambda(\lambda-1)\tilde{\Phi}]$. Put $\mathcal{S} = \mathrm{Spec}(R)$ and $\mathcal{S}_{\mathrm{ord}} = \mathrm{Spec}(R_{\mathrm{ord}})$. We choose once and for all a lift φ to $\mathcal{S}$ of the Frobenius morphism on $\mathbb{F}_p(\lambda)$ by defining $\varphi(\lambda) = \lambda^p$. We write $D(t, \rho^-)$ (resp. $D(t, \rho^+)$) for the open (resp. closed) rigid analytic disc with center t and radius ρ. Let k be the algebraic closure of $\mathbb{F}_p$. In this section, we denote by F the relative Frobenius morphism.

Lemma 1.3.1 *(a) We have that $H^1_{\mathrm{dR}}(Z/\mathcal{S}_{\mathrm{ord}})_{\tilde{\chi}}$ is an F-crystal.*

(b) We write $\mathrm{Fil}^1 = H^0(Z, \Omega^1)_{\tilde{\chi}}$ for the first part of the Hodge filtration. There exists an F-crystal $U \subset H^1_{\mathrm{dR}}(Z/\mathcal{S})_{\tilde{\chi}}$ such that

$$H^1_{\mathrm{dR}}(Z/\mathcal{S}_{\mathrm{ord}})_{\tilde{\chi}} = U \oplus \mathrm{Fil}^1.$$

The F-crystal U is called the unit root part.

Part (a) of the lemma holds without assumption on r and $b(p^i)$.

Proof: There exists a canonical isomorphism $H^1_{\mathrm{dR}}(Z/\mathcal{S}) \simeq H^1_{\mathrm{cris}}(\bar{Z}/\mathcal{S})$ ([3, Corollary 7.4]). This endows $H^1_{\mathrm{dR}}(Z/\mathcal{S})$ with the structure of an F-crystal. Concretely, this means that we have a horizontal morphism $F(\varphi) : \varphi^* H^1_{\mathrm{dR}}(Z/\mathcal{S}) \to H^1_{\mathrm{dR}}(Z/\mathcal{S})$ such that $F(\varphi) \otimes \mathbb{Q}$ is an isomorphism ([25, Definition 1.3]). The morphism $F(\varphi)$ depends on the lift φ we have chosen, but it suffices to consider $F(\varphi)$ for one chosen lift. Therefore we will sometimes drop φ from the notation.

Write R' for the p-adic completion of $W(\mathbb{F}_q)[\lambda][1/\lambda(\lambda-1)]$. Then

$$H^1_{\mathrm{dR}}(Z/\mathcal{S})_{\tilde{\chi}} \otimes_R R' = \oplus_{i=0}^{f-1} H^1_{\mathrm{dR}}(Z/\mathcal{S})_{\chi^{p^i}}.$$

Restricting to the eigenspaces, we have that $F(\varphi) : \varphi^* H^1_{\mathrm{dR}}(Z/\mathcal{S})_{\chi^{p^i}} \to H^1_{\mathrm{dR}}(Z/\mathcal{S})_{\chi^{p^{i+1}}}$. This proves (a).

It is proved in [32] that $F(\varphi)\mathrm{Fil}^1 \subset pH^1(Z/\mathcal{S})_{\tilde{\chi}}$. Lemma 1.2.1 states that for $t \in \mathcal{S}_{\mathrm{ord}}$ we have that $F(\varphi) : \varphi^* H^1(\bar{Z}_{\bar{t}}, \mathcal{O})_{\tilde{\chi}} \to H^1(\bar{Z}_{\bar{t}}, \mathcal{O})_{\tilde{\chi}}$ is an isomorphism. This implies that the Newton polygon of $t^* H^1_{\mathrm{dR}}(Z/\mathcal{S})_{\tilde{\chi}}$ is *ordinary*, that is, has f slopes zero and f slopes 1. Therefore (b) follows from [25, Theorem 4.1]. □

The goal of this section is to give a concrete description of the unit root crystal U, extending the description for $m = 2$ given in [25, Section 8]. In the rest of this section we suppose for simplicity that the dual character of χ does not equal χ^{p^i}, for some i. If $\chi = \chi^{p^i}$ Poincaré duality induces a symplectic pairing on $H^1_{\rm dR}(Z/S)_{\tilde{\chi}}$. This happens for example for $m = 2$, where χ is self dual. In this case one needs to make a small modification to the construction below. Namely, one should not choose $\alpha_t(0)$ and $\beta_t(0)$ independently, but should make sure that the chosen basis respects the symplectic structure. Since it is obvious how to adept the arguments, we leave out this case here.

We write $M := H^1_{\rm dR}(Z/\mathcal{S}_{\rm ord})_{\tilde{\chi}}$. Let $t \in W(\bar{\mathbb{F}}_p) - \{0, 1\}$ with $\Phi(t) \neq 0$. Write M_t for the restriction of M to the open rigid disk $D(t, 1-)$ and U_t for its unit root part. Both M_t and U_t are crystals over $W(k)[[\lambda - t]]$. Recall that we may write

$$M_t = \oplus_{i=0}^{f-1} M_t(i), \qquad U_t = \oplus_{i=0}^{f-1} U_t(i).$$

We write M_t^0 (resp. U_t^0) for the value at $\lambda = t$ of M_t (resp. U_t); these are crystals over $W(k)$. The special fiber $M_t^0(i) \otimes k$ admits a basis $\bar{\alpha}_t^0(i), \bar{\beta}_t^0(i)$ such that

$$F\sigma^* \bar{\alpha}_t^0(i-1) = \bar{\alpha}_t^0(i), \qquad \mathcal{C}\bar{\beta}_t^0(i) = \sigma^* \bar{\beta}_t^0(i-1).$$

Namely, it follows from the results of Section 1.2 that $F^f(\sigma^f)^*\xi(0) = \Phi(t)\xi(0)$. Therefore we may define $\bar{\alpha}_t^0(0)$ as a suitable multiple of $\xi(0)$ and put $\bar{\alpha}_t^0(i) = F^i(\sigma^i)^*\bar{\alpha}_t^0(0)$. Similarly, we may define $\bar{\beta}_t^0(0)$ such that $\mathcal{C}^f \bar{\beta}_t^0(0) = (\sigma^f)^*\bar{\beta}_t^0(0)$ and put $\bar{\beta}_t^0(i) = \mathcal{C}^{f-i}(\sigma^i)^*\bar{\beta}_t^0(0)$.

It is clear that the basis $\bar{\alpha}_t^0(i), \bar{\beta}_t^0(i)$ lifts to a horizontal basis $\alpha_t^0(i), \beta_t^0(i)$ of M_t^0 such that

$$F\varphi\alpha_t^0(i-1) = \alpha_t^0(i), \qquad F\varphi\beta_t^0(i-1) = p\beta_t^0(i). \tag{10}$$

Proposition 1.3.2 *The vectors $\alpha_t^0(i)$ and $\beta_t^0(i)$ are the value at t of a horizontal basis of M_t over $D(t, 1^-)$ which we denote by $\alpha_t(i), \beta_t(i)$.*

Proof: This is proved in [25, Proposition 3.12]. We sketch the argument.

Claim 1: We first claim that $\alpha_t^0(i)$ and $\beta_t^0(i)$ extend to a horizontal basis over $D(t, \rho_0^+)$, where ρ_0 is the valuation of $p^{1/(p-1)}$. The reason is that

$$(\lambda - t) \subset R_{\rho_0} = W(k)\{\{\rho_0^{-1}z\}\} = \{\sum a_n \left(\frac{z}{\rho_0}\right)^n \mid a_n \to 0\}$$

is a PD-ideal ([3, Example 3.2]). Therefore there exists an isomorphism $\iota^* M_t \xrightarrow{\sim} {\rm ev}^* M_t = M_t \otimes_{W(k)} R_{\rho_0}$. Here $\iota, {\rm ev} : W(k)[[\lambda - t]] \to R_{\rho_0}$ are the inclusion and evaluation at t, respectively. The horizontal basis $\alpha_t(i), \beta_t(i)$ over $D(t, \rho_0^+)$ is the basis on $\iota^* M_t$ corresponding to the basis $\alpha_t^0(i), \beta_t^0(i)$ of M_t^0 that we defined above. It follows from the construction that $F^f\varphi^f\alpha_t(0) = \alpha_t(0)$ and $F^f\varphi^f\beta_t(0) = q\beta_t(0)$, therefore we may define $\alpha_t(i) := F^i\varphi^i\alpha_t(0)$ and $\beta_t(i) := F^i\varphi^i\beta_t(0)/p^{-i}$.

Claim 2: Next we claim that $\alpha_t(i)$ and $\beta_t(i)$ extend to a horizontal basis over $D(t, \rho^+)$ for all $0 \leq \rho < 1$. Let K be the fraction field of R and put $K_\rho = R_\rho \otimes K$. Choose some basis of M_t and write A for the matrix of $F(\varphi) : \varphi^* M_t \to M_t$ with respect to this basis. The horizontal basis $\alpha_t(i), \beta_t(i)$ over $D(t, \rho_0^+)$ we constructed above defines a map $M_t^0 \otimes K_{\rho_0} \to M_t \otimes K_{\rho_0}$. Write Y for the matrix of this map. We obtain a commutative diagram

$$\begin{array}{ccc} \varphi^* M_t \otimes K_{\rho_0} & \xleftarrow{\varphi(Y)} & \varphi^* M_t^0 \otimes K_{\rho_0} \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A(\lambda=t)} \\ M_t \otimes K_{\rho_0} & \xleftarrow[Y]{} & M_t^0 \otimes K_{\rho_0}. \end{array}$$

Note that A has coefficients in $W(k)[[\lambda - t]]$. Therefore A converges and is bounded on the open disk $D(t, 1-)$. Since M_t is an F-crystal we have moreover that $A(\lambda = t) \otimes K$ is invertible. This implies that if $\varphi(Y)$ converges on $D(t,\rho^+)$ then Y converges on $D(t,\rho^+)$ also. Write $Y=\sum_j Y_j z^j$, where Y_i is a matrix with coefficients in K. Then $\varphi(Y)=\sum_j \sigma(Y_j)z^{pj}$. Hence if Y converges on $D(t,\rho^+)$ then $\varphi(Y)$ converges on $D(t,(\rho^{1/p})^+)$. Since we have already shown that Y converges on $D(t,\rho_0^+)$ it follows that Y converges on $D(t,1-)$. Therefore $\alpha_t(0)$ and $\beta_t(0)$ are the value at $\lambda=t$ of a basis of $M_t(0)$ over $D(t,1^-)$. We continue to denote this basis by $\alpha_t(0)$ and $\beta_t(0)$. It is clear from the construction that $\alpha_t(0)$ and $\beta_t(0)$ are horizontal and satisfy $F^f\alpha_t(0)=\alpha_t(0)$ and $F^f\beta_t(0)=q\beta_t(0)$. Therefore $\alpha_t(i)$ and $\beta_t(i)$ extend for $i=0,\ldots,f-1$.

Claim 3: We claim that $\alpha_t(0)$ is bounded on $D(t,1^+)$ ([25, Proposition 3.1.3]). We have already seen that $F^f\varphi^f\alpha_t(0)=\alpha_t(0)$. Moreover, $\varphi^{fn}(\alpha_t(0))\equiv\sigma^{fn}(\alpha_t^0(0)) \bmod (\lambda-t)^{p^{fn}}$. Therefore

$$\alpha_t(0)=\lim_{n\to\infty}F^f\circ\varphi^f(F^f)\circ\cdots\circ\varphi^{fn}(F^f)\varphi^{f(n+1)}(\alpha_t^0(0)).$$

This obviously converges. The same argument applies to $\alpha_t(i)$ for $i=0,\ldots,f-1$. □

Lemma 1.3.3 *Every bounded horizontal section of $M_t(0)$ is a multiple of $\alpha_t(0)$.*

Proof: This is proved as in [25, Corollary 7.5]. We sketch the argument.

Since $\alpha_t(0)$ and $\beta_t(0)$ form a basis of $M_t(0)$, there exists functions $f_t(0)$ and $g_t(0)$ such that the restriction of $\omega(0)$ to $D(t,1^-)$ can be written as $f_t(0)\alpha_t(0)+g_t(0)\beta_t(0)$. Since $\alpha_t(0)$ and $\beta_t(0)$ are horizontal and $\omega(0)$ satisfies the Picard–Fuchs differential equation (4), it follows that $f_t(0)$ and $g_0(t)$ are solutions of the Picard–Fuchs differential equation (regarded as ordinary differential equation). We put $\tau(0)=f_t(0)/g_t(0)\in W(k)[[\lambda-t]][1/p]$. It is called the *period*.

Recall that the Kodaira–Spencer map is defined as

$$\mathrm{Fil}^1M_t\overset{\nabla(\frac{\mathrm{d}}{\mathrm{d}\lambda})}{\longrightarrow}M_t\to M_t/\mathrm{Fil}^1M_t\simeq U_t.$$

Lemma 1.1.2 states that $\omega(i)$ and $\omega(i)'=\nabla(\frac{\mathrm{d}}{\mathrm{d}\lambda})\omega(i)$ are linearly independent for every $t\neq 0,1,\infty$. This implies that the Kodaira–Spencer map is nontrivial. The Kodaira–Spencer map sends $\eta\in\mathrm{Fil}^1M_t(0)$ to $\tau(0)'\eta$, where $\tau(0)'$ is the derivative of $\tau(0)$ with respect to λ ([25, Lemma 7.1]). The Kodaira–Spencer map is nontrivial if and only if the curve $Z\otimes k[t]/(t^2)$ is nonconstant. Using the congruences of [25, Proposition 7.4] it follows that there exists an unbounded solution of (4) in $D(t,1^-)$. This proves the lemma. □

Using our previous notation, we find that

$$\tau(0)'=\frac{f_t(0)'g_t(0)-f_t(0)g_t(0)'}{g_t(0)^2}.$$

Since $\tau(0)'\neq 0$, we conclude that $f_t(0)$ and $g_t(0)$ are linearly independent solutions of the differential equation at $\lambda=t$, i.e. they form a basis of solutions at t. It is not so easy to give the boundary conditions which determine the solutions $f_t(0)$ and $g_t(0)$, and hence $\tau(0)$. In [1, Section II.6.3.2] this is worked out for $m=2$ and $t=1/2$. (There are some mistakes in the formulas written there.) Compare to [15]??

Write

$$\alpha_t(i)=h_{1,t}(i)\lambda(\lambda-1)\omega(i)+h_{2,t}(i)\lambda(\lambda-1)\omega(i)'.$$

By assumption $\alpha_t(i)$ is horizontal. An elementary computation shows that this implies that $h_{2,t}(i)$ and $h_{1,t}(i)$ satisfy

$$h_{2,t}(i)''\lambda(\lambda-1)+h_{2,t}(i)'((A(i)+B(i)+1)\lambda-C(i))+h_{2,t}A(i)B(i)=0 \tag{11}$$

$$h_{1,t}(i)=h_{2,t}(i)\left(\frac{1-C(i)}{\lambda}+\frac{C(i)-A(i)-B(i)}{\lambda-1}\right)-h'_{2,t}(i). \tag{12}$$

In particular, $h_{2,t}(i)$ is a local solution to the differential equation dual to the Picard–Fuchs differential equation (4) of $M(i)$. We may suppose that the constant term of $h_{2,t}(i)$ is one. The existence of the unit root crystal is essentially equivalent to the following proposition.

Proposition 1.3.4 *There exist functions $H(i)$ and $G(i)$ in R_{ord} whose local expansions at t are $h_{2,t}(i)'/h_{2,t}(i)$ and $h_{2,t}(i)/h_{2,t}(i)^{\varphi}$, respectively.*

Proof: This follows from [25, 4.1.9]. The idea is the following. Note that the proof also proves the existence of the unit root crystal (Lemma 1.3.1.b).

Recall that $M(0)$ has a basis $\omega(0), \omega(0)'$. The matrix of $F^f \circ (\varphi^f)^*$ with respect to this basis may be written as

$$\begin{pmatrix} qA & C \\ qB & D \end{pmatrix}.$$

Since the special fiber of $M(0)$ is ordinary, it follows that d is invertible. To find a basis of the unit root crystal $U(0)$, we need to find an element $\eta(0) = E(0)\lambda(\lambda-1)\omega(0) + \lambda(\lambda-1)\omega(0)' \in M(0)$ such that the span of $\eta(0)$ is stabilized under $F^f \circ (\varphi^f)^*$. In other words, we want to find a basis such that the matrix of $F^f \circ (\varphi^f)^*$ is lower triangular. This amounts to finding $E(0) \in R_{\text{ord}}$ satisfying

$$E(0) = \frac{1}{D}\left(\frac{qA(\varphi^f)^*E(0) + C}{1 + qD^{-1}B(\varphi^f)^*E(0)}\right).$$

The function $E(0)$ can be shown to exist by approximating modulo higher and higher powers of p ([25, 4.1.7]).

We may define $\eta(i) = F^i(\varphi^i)^*\eta(0)$, and write

$$\eta(i) = E(i)\lambda(\lambda-1)\omega(i) + \lambda(\lambda-1)\omega(i)' \in M(i).$$

Then U is spanned over R_{ord} by $\eta(0), \ldots, \eta(f-1)$. Now U is an F-crystal if and only if U is preserved by the Gauß–Manin connection ∇. It suffices to check this over $W(k)[[\lambda - t]]$ for $t \in \mathcal{S}_{\text{ord}}$.

We have already shown that over $W(k)[[\lambda - t]]$ there exist horizontal vectors

$$\alpha_t(i) = h_{1,t}(i)\lambda(\lambda-1)\omega(i) + h_{2,t}(i)\lambda(\lambda-1)\omega(i)'.$$

Since $E(i)$ is unique, it follows that $h_{1,t}(i)/h_{2,t}(i)$ is the power series expansion at t of $E(i)$. Note

$$\frac{h_{1,t}(i)}{h_{2,t}(i)} = \frac{1 - C(i)}{\lambda} + \frac{C(i) - A(i) - B(i)}{\lambda - 1} - \frac{h_{2,t}(i)'}{h_{2,t}(i)}.$$

Therefore

$$H(i) := \frac{1 - C(i)}{\lambda} + \frac{C(i) - A(i) - B(i)}{\lambda - 1} - E(i)$$

has power series expansion $h_{2,t}(i)'/h_{2,t}(i)$ at t. This shows the existence of $H(i)$. Locally at t, we have that $\eta(i)$ equals $\alpha_t(i)/h_{2,t}(i)$. Therefore $\nabla\eta(i) = -H(i)\eta(i) \otimes \mathrm{d}\lambda$. This proves that U is an F-crystal.

Since U is an F-crystal, the following diagram commutes

$$\begin{array}{ccc} \varphi^*U & \xrightarrow{F} & U \\ \varphi^*\nabla \downarrow & & \downarrow \nabla \\ \varphi^*U \otimes \Omega^1_{\mathcal{S}_{\text{ord}}} & \xrightarrow{F\otimes \text{Id}} & U \otimes \Omega^1_{\mathcal{S}_{\text{ord}}}. \end{array} \tag{13}$$

Write $F\varphi^*\eta(i-1) = G(i)\eta(i)$. It is obvious that $G(i) \in R_{\text{ord}}$ exits. Then

$$\nabla \circ F\varphi^*\eta(i-1) = (G(i)' - H(i)G(i))\eta(i) \otimes \mathrm{d}\lambda.$$

One computes that

$$(F\otimes \mathrm{Id})\circ \varphi^*\nabla\varphi^*\eta(i-1) = -p\lambda^{p-1}H(i-1)^{\varphi}G(i)\eta(i)\otimes \mathrm{d}\lambda.$$

This implies that

$$\frac{G'(i)}{G(i)} = H(i) - p\lambda^{p-1}H(i-1)^{\varphi}.$$

The local expansion of $G(i)$ at t is $h_{2,t}(i)/(h_{2,t}(i-1))^{\varphi}$. One checks that $G(i)$ is a solution to the dual differential equation (11) modulo p.

Lemma 1.3.5 *We have*

$$G(i)\equiv \Phi(i) \pmod{p}.$$

Proof: We want to compute the image of $\eta(i)$ in $H^1(\bar{Z},\mathcal{O})_{\chi^{p^i}}$. Since $\eta(i) = E(i)\lambda(\lambda-1)\omega(i) + \lambda(\lambda-1)\omega(i)'$, it suffices to compute the image of $\lambda(\lambda-1)\omega(i)'$. It is explained in Lemma 1.1.3.(b) how to do this.

First we need to write $\omega(i)' = (1-\sigma_3(i))\mathrm{d}x/x^{1-\sigma_1(p^i)}(x-1)^{1-\sigma_2(p^i)}(x-\lambda)^{2-\sigma_3(p^i)}$ in terms of the basis (Lemma 1.1.2). To do this, note that

$$\lambda(\lambda-1)\omega(i)' \quad + \quad \mathrm{d}\frac{x^{\sigma_1(p^i)}(x-1)^{\sigma_2(p^i)}}{(x-\lambda)^{1-\sigma_3(p^i)}} = \tag{14}$$

$$= \quad \frac{(1-\sigma_4(p^i))x+\lambda(\sigma_3(p^i)-1)+1-\sigma_1(p^i)-\sigma_3(p^i)\mathrm{d}x}{x^{1-\sigma_1(p^i)}(x-1)^{1-\sigma_2(p^i)}(x-\lambda)^{1-\sigma_3(p^i)}} =: \tilde{\omega}(i)'. \tag{15}$$

Recall that

$$\xi(i) = \frac{z^{p^i}}{x^{1+[p^i\sigma_1]}(x-1)^{[p^i\sigma_2]}(x-\lambda)^{[p^i\sigma_3]}} = \frac{(x-1)^{\sigma_2(p^i)}(x-\lambda)^{\sigma_3(p^i)}}{x^{1-\sigma_1(p^i)}}$$

is a basis for $H^1(\bar{Z},\mathcal{O})_{\chi^{p^i}}$. Note that $\tilde{\omega}(i)+\mathrm{d}\xi(i)$ is holomorphic outside 0. Lemma 1.1.3.b implies that the image of $\lambda(\lambda-1)\omega(i)'$ is $\xi(i)$. The lemma follows now from the definition of $\Phi(i)$. □

Lemma 1.3.5 implies that $h_{2,t}(i)\equiv\Phi(i)h_{2,t}(i-1)^{\varphi}$ mod p. Therefore

$$H(i) = \frac{h_{2,t}(i)'}{h_{2,t}(i)} \equiv \frac{\Phi(i)'}{\Phi(i)} \pmod{p}.$$

Here we use that the derivative of $h_{2,t}(i-1)^{\varphi}(\lambda) = h_{2,t}(i-1)(\lambda^p)$ is zero modulo p.

Proposition 1.3.6 *For every $n\geq 1$ we define functions $B_n(i)\in k[\lambda]$ by*

$$B_n(i) = (-1)^{N_n(i)}\sum_{j_1+j_2=N_n(i)}\binom{[n\sigma_2(p^{i-1})]}{j_1}\binom{[n\sigma_3(p^{i-1})}{j_2}\lambda^{j_2},$$

where $N_n(i) = n-1-[n\sigma_4(p^{i-1})]$. We have

$$G(i) = \lim_{n\to\infty}\frac{B_n(i)}{B_{n-1}^{\varphi}}(i-1).$$

Proof: This follows from the result of [45]. In this paper it is shown how to compute $G(i)$, by using an identification of U with the Witt vector cohomology group $H^1(Z,\mathcal{W}_p\mathcal{O})_{\tilde{\chi}}$. (We refer to [45] for the definition of this cohomology group.) The proposition is a special case of [45, Section 5.4]. □

Since $(B_n(i-1)^\varphi)^{-1}B_{n+1}(i) \equiv (B_{n-1}(i-1)^\varphi)^{-1}B_n(i) \bmod p^n$ for all $n \geq 1$, we have that

$$G(i) \equiv \frac{B_n(i)}{B_{n-1}(i-1)^\varphi} \pmod{p^n}.$$

As in (6), one checks that the polynomials $B_n(i)$ are equivalent modulo p^n to certain expansion coefficients of $\omega(i)$. Therefore it follows from [27] that $B_n(i)$ is a solution modulo p^n of the Picard–Fuchs differential equation. This is also noted in [45, Example 5.5].

Remark 1.3.7 If $m = 2$ it is easy to express the function $G(0)$ in terms of Gauß' hypergeometric function $F(1/2, 1/2, 1; \lambda)$. Namely,

$$G(0)(\lambda) = (-1)^{(p-1)/2}\frac{F(\frac{1}{2},\frac{1}{2},1;\lambda)}{F(\frac{1}{2},\frac{1}{2},1;\lambda^p)} \in \mathbb{Z}[\frac{1}{2}][[\lambda]].$$

This follows easily from Proposition 1.3.6. It should not be too difficult to generalize this to arbitrary m.

As in [25], this description of the unit root crystal allows us to compute the χ-part of the zeta function of $\bar{Z}$. Choose $t \in W(\bar{\mathbb{F}}_p)$ with $\Phi(t) \neq 0$. As in [26, Section I], we write $P_{1,\chi} = \det(1 - TF^f)$ for the characteristic polynomial of the fth power of the (relative) Frobenius morphism on $H^1_{\mathrm{dR}}(Z_t)_\chi$. Suppose that $t \in W(\mathbb{F}_{q^n})$ with $q = p^f$. Define

$$\begin{aligned} \mathcal{G}(0)(\lambda) &:= G(1)(\lambda^{f-1}) \circ G(2)(\lambda^{f-2}) \circ \cdots \circ G(0)(\lambda), \\ \mathcal{G}_n(0)(\lambda) &:= \mathcal{G}(0)(\lambda) \circ \mathcal{G}(0)(\lambda^q) \circ \cdots \circ \mathcal{G}(0)(\lambda^{q^{n-1}}). \end{aligned}$$

Then $F^{fn}(\varphi)^{fn}\eta(0) = \mathcal{G}_n(0)\eta(0)$.

Proposition 1.3.8 *We have*

$$P_{1,\chi} = (1 - \mathcal{G}_n(0)T)(1 - \frac{q}{\mathcal{G}_n(0)}T).$$

Proof: This follows from the above discussion. □

1.4 The supersingular polynomial In this section, we apply the results of Section 1.3 to obtain an expression of the so called supersingular polynomial. We assume that $p > 3$. Recall that

$$\mathrm{ss}_p(j) = \prod (j - j(E)) \in \mathbb{F}_p[j],$$

where the product is taken over the elliptic curves $E/\bar{\mathbb{F}}_p$ which are supersingular. Denote

$$\alpha = \left[\frac{p}{12}\right], \quad \delta = \begin{cases} 0 & \text{if } p \equiv 1 \mod 3, \\ 1 & \text{if } p \equiv 2 \mod 3, \end{cases} \qquad \epsilon = \begin{cases} 0 & \text{if } p \equiv 1 \mod 4, \\ 1 & \text{if } p \equiv 3 \mod 4. \end{cases}$$

There exists a polynomial $\tilde{\mathrm{ss}}_p(j)$ with

$$\mathrm{ss}_p(j) = j^\delta (j - 1728)^\epsilon \tilde{\mathrm{ss}}_p(j),$$

compare to [21, Section 2]. The polynomial $\tilde{\mathrm{ss}}_p(j)$ does not have a zero at 0 and 1728.

Write $m = 12$ and $S = \mathrm{Spec}(\mathbb{Z}_p[\lambda, \zeta_m, 1/m\lambda(1-\lambda)])$, where $\zeta_m \in \mathbb{Q}_p^{\mathrm{nr}}$ is a primitive 12th root of unity. Choose an irreducible character $\chi : \mathbb{Z}/m \to \mathbb{Q}_p(\zeta_m)$. As before, we let f be the order of p in $(\mathbb{Z}/m)^*$, and put $\tilde{\chi} = \chi + \chi^p + \cdots + \chi^{p^{f-1}}$.

Let $Z \to \mathbb{P}^1_S$ be the m-cyclic cover of type $(1, 11, 5, 7)$ given by

$$z^{12} = x(x-1)^{11}(x-\lambda)^5, \qquad (x, z) \mapsto x.$$

Define $H/\mathbb{Q}$ to be the Hurwitz space parameterizing Galois covers $g : Y \to \mathbb{P}^1_{\mathbb{Q}}$ with Galois group $(\mathbb{Z}/p)^f \rtimes_{\tilde{\chi}} \mathbb{Z}/m$ which are only branched at $0, 1, \lambda, \infty$. We suppose moreover that g factors as $Y \to Z \to \mathbb{P}^1$, with $Z \to \mathbb{P}^1$ as above and $Y \to Z$ étale. Write $\pi : \mathcal{H} \to \mathbb{P}^1_\lambda$ for the map with sends g to λ and write $\tilde{\pi} : \tilde{\mathcal{H}} \to \mathbb{P}^1_\lambda$ for the Galois closure of π.

It is shown in [8] that $\tilde{\pi}$ is an $\mathrm{PSL}_2(p)$-Galois cover branched only at $0, 1, \infty$ of order 2,3,p. The cover $\tilde{\pi}$ is *rigid* ([43, Proposition 7.4.2]). In this particular case, this means that there exists a unique $\mathrm{PSL}_2(p)$-Galois of the projective line branched at three points of order 2,3,p. (We refer the reader to [43, Chapter 7] for a more precise statement and the definition of rigidity.) We conclude that there exists a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{H}} \otimes \bar{\mathbb{Q}} & \stackrel{\sim}{\to} & X(p) \\ \downarrow & & \downarrow \\ \mathbb{P}^1_\lambda & \stackrel{\sim}{\to} & \mathbb{P}^1_j. \end{array}$$

Here $X(p)/\bar{\mathbb{Q}}$ is the modular curve parameterizing elliptic curves E with full level p structure and $X(p) \to \mathbb{P}^1_j$ sends E to its j-invariant. The arrow $\mathbb{P}^1_\lambda \to \mathbb{P}^1_j$ is given by $\lambda \mapsto 1728\lambda =: j$.

Lemma 1.4.1 *Write $J(Z)^{\mathrm{new}}$ for the new part of the Jacobian of Z. Then there exists an elliptic curve E_λ such that*

$$J(Z)^{\mathrm{new}} \sim E_\lambda^4.$$

Proof: We first note that $\mathrm{Aut}(Z) = \mathbb{Z}/12 \rtimes (\mathbb{Z}/12)^*$. We may choose generators of this group such that

$$\psi(x,z) = (x, \zeta_{12} z), \quad \tau_5(x) = \frac{x-\lambda}{x-1}, \quad \tau_7(x) = \frac{\lambda}{x}, \quad \tau_{11}(x) = \lambda\frac{x-1}{x-\lambda}.$$

One checks that $\tau_i \psi \tau_i = \varphi^i$. Write Z_6 (resp. Z_4) for the quotient of Z by ψ^2 (resp. ψ^3). The new part J^{new} of the Jacobian $J = J(Z)$ of Z is defined as the quotient of J by the image of $J(Z_6) \cup J(Z_4)$. Then $V = \langle \tau_5, \tau_7 \rangle$ acts on J^{new} and $E_\lambda := J^{\mathrm{new}}/V$ is an elliptic curve. The statement of the lemma follows from the fact that ψ acts on J^{new}. □

Let $\bar{Z} := Z \otimes \mathbb{F}_p$. For $i = 1, 5, 7, 11$, Lemma 1.1.2 implies that $\dim H^1(\bar{Z}, \mathcal{O})_{\chi^i} = 1$. We denote by

$$\xi_i = \frac{z^i}{x} x^{-[p\frac{1}{12}]}(x-1)^{-[p\frac{11}{12}]}(x-\lambda)^{-[p\frac{5}{12}]}$$

a basis vector of $\dim H^1(\bar{Z}, \mathcal{O})_{\chi^i}$, as defined in Lemma 1.1.3. (Note that the numbering is different from what we wrote before.) Define polynomials $\Phi(i) \in \mathbb{F}_p[\lambda]$ by

$$F\xi_i = \Phi(i)\xi_{pi}.$$

Lemma 1.4.2 *Write $\gamma_0 = [p\frac{5}{12}]$, $\gamma_1 = [p\frac{11}{12}]$. Then*

(a)

$$\Phi(1) = (\lambda-1)^{\gamma_1}\Phi(5), \quad \Phi(7) = \lambda^{\gamma_0}(\lambda-1)^{\gamma_1}\Phi(5), \quad \Phi(11) = \lambda^{\gamma_1}\Phi(5).$$

(b) The polynomial $\Phi(5)$ does not have a zero at $0, 1$.

Proof: This is an elementary computation using Lemma 1.2.3. One also uses the contiguity relations for the hypergeometric functions ([52]). □

Proposition 1.4.3 *Suppose $p > 3$, then* $\Phi(5) \equiv \tilde{\mathrm{ss}}_p \bmod p$.

Proof: The fact that $X(p)$ is isomorphic to $\mathcal{H}$ should imply that $j(E_\lambda)$ equals λ (up to some constant in $\bar{\mathbb{Q}}$.) I did not manage to verify this directly. A less direct proof is given in [13], using the stable reduction of $\mathcal{H}$ and $X(p)$. In [13] it is assumed that $p \equiv 1 \bmod 12$, but one can get rid of this assumption. Details will be worked out in a later chapter. □

The above formula is a mod p version of one of the formulas for the supersingular polynomial obtained in [21]. The other formulas found in that paper seem to have no interpretation in our setting.

2 Generalities on deformation data

In this section we define deformation data (Section 2.1), and explain their relation to stable reduction of Galois covers of curves (Section 2.2). Section 2.3 introduces Galois covers with special reduction which plays a key role in the rest of the paper. These are Galois covers with bad reduction such that the corresponding deformation datum is special. This definition is an extension to the case of covers of the projective line branched at more than three points of the definition of [49]. We introduce the Hasse invariant Φ_*, and show that it is nonzero in the special case. In Section 2.4 we show a lifting lemma. Roughly speaking, the statement is that every deformation datum comes from the stable reduction of a Galois cover with bad reduction.

2.1 Definitions Let k be an algebraically closed field of characteristic $p > 2$. Let H be a finite group of order prime to p. Fix a character $\chi : H \to \mathbb{F}_p^\times$.

Definition 2.1.1 A *deformation datum* of type (H, χ) is a pair (g, ω), where $g : Z_k \to X_k = \mathbb{P}^1_k$ is an H-Galois cover and ω is a meromorphic differential form on Z_k such that the following conditions hold.

(a) We have
$$\beta^*\omega = \chi(\beta) \cdot \omega, \qquad \text{for all } \beta \in H. \tag{16}$$

(b) The differential ω is either logarithmic (i.e. $\omega = \mathrm{d}u/u$) or exact (i.e. $\omega = \mathrm{d}u$). If ω is exact we assume moreover that ω is holomorphic. In the first case, the deformation datum (g, ω) is called *multiplicative*. In the second case it is called *additive*.

Let (g, ω) is a deformation datum. For each closed point $x \in X_k$ we define the following invariants.
$$m_x := |H_z|, \qquad h_x := \mathrm{ord}_z(\omega) + 1, \qquad \sigma_x := h_x/m_x.$$
Here $z \in Z_k$ is some point above x and $H_z \subset H$ is the stabilizer of z. A point x with $(m_x, h_x) \neq (1, 1)$ is called a critical point of the deformation datum. We denote by $(\tau_i)_{i\in\mathbb{B}}$ the set of critical points of ω which we call *tails*. Define $\mathbb{B}_{\mathrm{wild}} = \{i \in \mathbb{B} \mid h_i = 0\}$; it is called the set of *wild tails*. This terminology is explained in Section 2.2. We denote by $\mathbb{B}' = \{i \in \mathbb{B} \mid \tau_i \neq \infty\}$.

We call $(\sigma_i)_{i\in\mathbb{B}}$ the *signature* of the Fuchsian deformation datum. Define a_i, ν_i by
$$\sigma_i = \frac{a_i}{p-1} + \nu_i, \qquad \text{where } 0 \leq a_i < p - 1 \quad \text{and } \nu_i \in \mathbb{Z}_{\geq 0}. \tag{17}$$

Example 2.1.2 Let $g : Z \to \mathbb{P}^1$ be an m-cyclic cover defined over k branched at $0, 1, \lambda, \infty$ of type $\mathbf{a} = (a_1, a_2, a_3, a_0)$ (Section 1.1). Suppose that $\sum a_i = 2m$. This implies that the k-dimension of $H^0(Z, \Omega)_\chi$ is one. Let $\omega = \omega_1^1$ be the basis of this space defined in Lemma 1.1.2. Recall that $\mathcal{C}\omega = \Phi_*^{(1/p)}\omega$, where $\Phi_* = \Phi_0^*$ is the Hasse invariant (Section 1.2). Suppose that $\Phi_*(\lambda) \neq 0$; we checked in Section 1.2 that this holds for general λ. Then $\Phi_*^{1/(p-1)}\omega$ is logarithmic. The invariants a_i defined above are the type (Notation 1.1.1) multiplied by $\gcd(p-1, a_0, a_1, a_2, a_3)$.

Recall that $g : Z_k \to \mathbb{P}^1_k$ is an H-Galois cover, where H is a group of order prime to p. Dividing out by the kernel of χ, we obtain a cyclic cover $g' : Z'_k \to \mathbb{P}^1_k$ of order dividing $p-1$. Since the induced character $\chi : H/\ker(\chi) \to \mathbb{F}_p^\times$ is injective, we may regard Z'_k as a connected component of the smooth projective curve given by the Kummer equation

$$z^{p-1} = \prod_{i \in \mathbb{B}'} (x - \tau_i)^{a_i}, \tag{18}$$

Note that ω descents to a differential form on Z'_k which we denote again by ω

In case $\mathbb{B}_{\rm wild} = \emptyset$, the differential ω is holomorphic, that is $\omega \in H^0(Z'_k, \Omega)_\chi$. We have seen that $H^0(Z'_k, \Omega)_\chi$ has dimension $|\mathbb{B}| - (\sum_i a_i)/(p-1)$ (equation (20)). If $\mathbb{B}_{\rm wild} \neq \emptyset$, the differential ω has logarithmic poles at τ_i for $i \in \mathbb{B}_{\rm wild}$. Lemma 2.1.3 implies that this dimension formula remains true. We denote by $H^0(Z'_k, \Omega^{\log})$ the space of meromorphic differentials which have at most logarithmic poles at $(\tau_i)_{i \in \mathbb{B}_{\rm wild}}$ and are holomorphic elsewhere.

Lemma 2.1.3 *The differentials*

$$\omega_j = \frac{x^{j-1} z \mathrm{d}x}{\prod_{i \in \mathbb{B}'}(x - \tau_i)}, \qquad j = 1, \ldots |\mathbb{B}_{\rm new}| + r - (\sum_i a_i)/(p-1),$$

form a basis of $H^0(Z'_k, \Omega^{\log})_\chi$.

Proof: This is proved like Lemma 1.1.2. □

2.2 Stable reduction We start by recalling some results on the stable reduction of Galois covers. This gives a natural way of producing deformation data and motivates the definition in the previous section. Let R be a complete discrete valuation ring with fraction field K of characteristic zero and residue field an algebraically closed field k of characteristic p. Let G be a finite group whose order is strictly divisible by p and let $f : Y \to X = \mathbb{P}^1_K$ be a G-Galois cover branched at $r+1 \geq 3$ points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3, \ldots, x_r$. After replacing K by a finite extension, we may assume that the x_i are K-rational. In this paper we assume that there exists a model $X_{0,R} = \mathbb{P}^1_R$ of X over R such that the x_i extend to pairwise disjoint sections $\mathrm{Spec}(R) \to X_{0,R}$. In other words, we assume that $(X; x_i)$ has good reduction.

Denote the ramification points of f by $y_1, \ldots, y_s$. We consider (y_i) as a marking on Y. After replacing K by a finite extension, there exists a unique extension $(Y_R; y_i)$ of $(Y; y_i)$ to a stably marked curve over R. The action of G extends to Y_R; write X_R for the quotient of Y_R by G. The map $f_R : Y_R \to X_R$ is called the *stable model* of f; its special fiber $\bar{f} : \bar{Y} \to \bar{X}$ is called the *stable reduction* of f [50, Definition 1.1]. The natural map $X_R \to X_{0,R}$ is an isomorphism on a unique irreducible component of $\bar{X} := X_R \otimes k$. We denote this component by $\bar{X}_0$ and call it the *original component*. All other irreducible components of $\bar{X}$ are contracted to a point.

We say that f has *good reduction* if $\bar{f}$ is separable. This is equivalent to $\bar{X}$ being smooth. If f does not have good reduction, we say it has *bad reduction*.

Suppose that f has bad reduction. Let T' be the dual graph of X. The set of vertices $\mathbb{V}'$ of T' corresponds to the irreducible components of $\bar{X}$. We denote by $\bar{X}_v$ the irreducible component corresponding to $v \in \mathbb{V}'$. The set $\mathbb{E}'$ of (oriented) edges of T' corresponds to the singularities of $\bar{X}$. If $e \in \mathbb{E}'$ is an edge with source v and target w, we denote by $\tau_e \in \bar{X}$ the corresponding point of intersection of $\bar{X}_v$ and $\bar{X}_w$. Let $v_0 \in \mathbb{V}'$ correspond to the original components $\bar{X}_0$. Write $\mathbb{B}_{\rm wild} \subset \{1, \ldots, r\}$ for the set indexing the branch points x_i of f whose ramification index is divisible by p. We define a graph T with vertices $\mathbb{V} = \mathbb{V}' \cup \mathbb{B}_{\rm wild}$. For every $i \in \mathbb{B}_{\rm wild}$, we also add one edge e_i whose source is the component to which x_i specializes and whose target is $i \in \mathbb{V}$, together with the opposite edge.

We consider the graph T to be oriented from v_0. An vertex $v \in V - \{v_0\}$ is called a *tail* if there is a unique edge with target v. We write $\mathbb{B} \subset \mathbb{V}$ for the set of tails and $\mathbb{I} = \mathbb{V} - \mathbb{B}$ for the complement. The vertices $v \in \mathbb{I}$ are called the *interior vertices*. It is proved in [40] that $v \in \mathbb{V} - \mathbb{B}_{\rm wild}$ is a tail if and only if the restriction of $\bar{f}$ to $\bar{X}_v$ is separable. (This is no longer true if one drops the assumption that p strictly divides the order of G.) A tail $\bar{X}_v$ is called *primitive* if one of the branch points x_i of f specializes to $\bar{X}_v$. Otherwise, the tail is called *new*. We write $\mathbb{B}_{\rm prim}$ (resp. $\mathbb{B}_{\rm new}$) for the set of primitive (resp. new) tails. Note that $\mathbb{B}_{\rm wild} \subset \mathbb{B}_{\rm prim}$.

Definition 2.2.1 We say that $\bar{X}$ is a *comb* if $\mathbb{V} = \mathbb{B} \cup \{v_0\}$.

If $\bar{X}$ is a comb, $\bar{f}$ is inseparable only over the original component $\bar{X}_0$.

Write $T_{\bar{Y}}$ for the graph corresponding to $\bar{Y}$ which is defined in the same way as the graph T. Choose a connected component $T^i_{\bar{Y}}$ of $\bar{f}^{-1}(\mathbb{I} \cup \mathbb{B}_{\rm wild}) \subset T_{\bar{Y}}$. Denote the fixed group of $T^i_{\bar{Y}}$ by G_0. Let $v \in \mathbb{I}$ be an interior vertex. The restriction of $\bar{f}$ to $\bar{X}_v$ is inseparable. Choose a component $\bar{Y}_v$ of $\bar{Y}$ above $\bar{X}_v$, where we assume that the vertex of $T_{\bar{Y}}$ corresponding to $\bar{Y}_v$ is contained in $T^i_{\bar{Y}}$. Let $G_v \subset G_0$ denote the decomposition group of the component $\bar{Y}_v$. It is easy to see that the inertia group of $\bar{Y}_v$ is independent of v ([40]). We denote this inertia group by I_0. It follows that $I_0 \lhd G_0$. Put $H_0 = G_0/I_0$.

The restriction of $\bar{f}$ to $\bar{Y}_v$ factors as $\bar{Y}_v \to \bar{Z}_v \to \bar{X}_v$, with $\bar{g}_v : \bar{Z}_v \to \bar{X}_v$ a separable Galois cover of order prime-to-p and $\bar{Y}_v \to \bar{Z}_v$ purely inseparable of degree p. The inseparable map $\bar{Y}_v \to \bar{Z}_v$ is generically endowed with the structure of a $\boldsymbol{\mu}_p$-torsor or an $\boldsymbol{\alpha}_p$-torsor. This structure is encoded in a meromorphic differential ω_v. Define $H_v := \mathrm{Gal}(\bar{Z}_v, \bar{X}_v)$; then G_v is the semi-direct product $I_0 \rtimes H_v$. The action of H_v on I_0 by conjugation gives rise to a character $\chi_v : H_v \to \mathbb{F}_p^\times$. This implies that

$$\beta^*\omega_v = \chi_v(\beta) \cdot \omega_v, \qquad \text{for all } \beta \in H_v.$$

We recall from [40] and [50] the existence of the *auxiliary cover* $f_{{\rm aux},R} : Y_{{\rm aux},R} \to X_R$. The auxiliary cover is a G_0-Galois cover over R with bad reduction to characteristic p. It is essentially characterized by the property that the restriction of $f_{{\rm aux},R}$ to the interior coincides with restriction of $\bar{f}$ corresponding to $\cup_{w \in T^i_{\bar{Y}}} \bar{Y}_w$. See Section 5.2 for more details. We define $Z_R = Y_{{\rm aux},R}/I_0$. Let $g_R : Z_R \to X_R$ be the corresponding H-Galois cover. We write $\bar{g} : \bar{Z} \to \bar{X}$ for its special fiber.

The original components $\bar{X}_0$ plays an essential role. We denote by ω the differential corresponding to $v = v_0$ and $\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0$ the restriction of $\bar{g}$ to $\bar{X}_0$.

Definition 2.2.2 We call $(\bar{g}_0, \omega)$ the *deformation datum* of f.

Note that the deformation datum depends on the choice of $\bar{Y}_0$. We omit this from the notation.

Let $\xi \in \bar{Z}_0$ be a closed point and τ its image in $\mathbb{P}^1_k$. Denote by H_ξ the stabilizer of ξ in H. Define

$$m_\tau := |H_\xi|, \qquad h_\tau := \mathrm{ord}_\xi(\omega) + 1, \qquad \sigma_\tau = h_\tau/m_\tau. \tag{19}$$

We say that τ is a *critical point* of the differential ω if $(m_\xi, h_\xi) \neq (1,1)$. Let (τ_i) be the critical points of ω.

Lemma 2.2.3 *The set of critical points of ω is contained in the set of edges with source v_0.*

Proof: This follows from [50, Proposition 1.7]. □

If $\bar{X}$ is a comb, Lemma 2.2.3 allows to simplify our notation. For $i \in \mathbb{B}$, there exists a unique edge e with source v_0 and target i. We denote by τ_i the corresponding point of $\bar{X}_0$. Lemma 2.2.3 implies that $(\tau_i)_{i \in \mathbb{B}}$ is exactly the set of critical points of ω. We then write m_i, h_i, σ_i instead of $m_{\tau_i}, h_{\tau_i}, \sigma_{\tau_i}$. For every i, choose a point $\xi_i \in \bar{Z}_0$ above τ_i and write $H(\xi_i) \subset H_0$ for its stabilizer. The set $\mathbb{B}_{\rm wild}$ is exactly $\{i \in \mathbb{B} \,|\, h_i = 0\}$.

Lemma 2.2.4 *Suppose that $\bar{X}$ is a comb. The σ_i satisfy the following properties:*

(a) $\sum_{i\in\mathbb{B}_{\rm prim}}\sigma_i+\sum_{i\in\mathbb{B}_{\rm new}}(\sigma_i-1)=r-1$,

(b) $\sigma_i>1$ *for all* $i\in\mathbb{B}_{\rm new}$.

Proof: This is proved in [40], see also [50, Corollary 1.11]. □

For every $i\in\mathbb{B}$, we define integers $0\le a_i<p-1$ uniquely characterized by the property (17).

The assumption that $(X;x_i)$ has good reduction implies that there is a one to one correspondence between the points x_i and the elements of $\mathbb{B}_{\rm prim}$. Therefore we may write $\mathbb{B}_{\rm prim}=\{0,\dots,r\}$. We may assume that $\tau_0=\infty$. This implies that $\tau_i\neq\infty$ for $i\in\mathbb{B}_{\rm new}$. Write $\mathbb{B}'_{\rm prim}=\{i\in\mathbb{B}_{\rm prim}\,|\,\tau_i\neq\infty\}$. We call $(\sigma_i)_{i\in\mathbb{B}}$ the *signature* of the differential ω.

Recall that $\bar{g}_0:\bar{Z}_0\to\bar{X}_0$ is an H_0-Galois cover, where H_0 is an abelian group of order prime to p. Let $\bar{g}'_0:\bar{Z}'_0\to\bar{X}_0$ be the quotient of $\bar{g}$ by the kernel of $\chi:H_0\to\mathbb{F}_p^\times$. Let H'_0 be the Galois group of $\bar{g}'_0$ and denote its order by m. We may regard H'_0 as a subgroup of $\mathbb{F}_p^\times$, via the character χ.

Lemma 2.2.5 *The disconnected cover*

$$\bar{g}_0^{\rm n}:\bar{Z}_0^{\rm n}:={\rm Ind}_{H'_0}^{\mathbb{F}_p^\times}\bar{Z}'_0\to\bar{X}_0$$

is given by

$$z^{(p-1)}=\prod_{i\in\mathbb{B}'}(x-\tau_i)^{a_i},\quad (x,z)\mapsto x.$$

The Galois action is given by $\beta^*(z)=\chi_0(\beta)\cdot z$ for $\beta\in H'_0$.

Proof: Kummer theory implies that there exist integers $(c_i)_{i\in\mathbb{B}}$ with $0<c_i<p-1$ and $\sum_{i\in\mathbb{B}}c_i\equiv 0 \bmod (p-1)$ such that $\bar{Z}''_0$ is the complete nonsingular curve associated to the equation

$$z^{(p-1)}=\prod_{i\in\mathbb{B}'_{\rm prim}}(x-\tau_i)^{c_i}\prod_{i\in\mathbb{B}_{\rm new}}(x-\tau_i)^{c_i}.$$

In [50], it is shown that $c_i\equiv a_i \bmod p-1$. □

2.3 Covers with special reduction In this section we define the concept of special reduction. The notation is as in Section 2.2.

Let $f:Y\to\mathbb{P}^1$ be a G-Galois cover defined over K branched at $r+1$ points $x_0,x_1,\dots x_r$ of order prime to p. As in Section 2.2, we suppose that f has bad reduction to characteristic p, but $(X;x_i)$ has good reduction. Recall that the deformation datum associated to f consists of a H_0-Galois cover $\bar{g}_0:\bar{Z}_0\to\bar{X}_0$ of order prime to p together with a differential ω on $\bar{Z}_0$ and a character $\chi_0:H_0\to\mathbb{F}_p^\times$. In this section, we suppose that this character is injective and that H is cyclic. We write (σ_i) for the signature of ω and (τ_i) for its set of critical points. Recall from Section 2.2 that we may write

$${\rm ord}_{z_i}(\omega)+1=\sigma_i=\frac{a_i}{p-1}+\nu_i,$$

where $\nu_i\ge1$ for all $i\in\mathbb{B}_{\rm new}$ and $0\le a_i<p-1$. Here z_i is some point of $\bar{Z}_0$ above τ_i. In the rest of this section, we make the following assumption.

Definition 2.3.1 We say that the deformation datum $(\bar{g}_0,\bar{\omega}_0)$ is *special* if $\nu_i=0$ for $i\in\mathbb{B}_{\rm prim}$ and $\nu_i=1$ for $i\in\mathbb{B}_{\rm new}$. If $\nu_i=1$ we require moreover that $a_i\neq0$.

In particular specialty implies that $\bar{X}$ is a comb. The condition that $a_i \neq 0$ if $\nu_i = 1$ is equivalent to $\sigma_i \neq p/(p-1)$. The reason for this condition is the following. The fact that $\bar{X}$ is a comb implies that the invariant $\sigma_i = h_i/m_i$ is the *ramification invariant* of the separable Galois cover $\bar{f}_i : \bar{Y}_i \to \bar{X}_i$ at an intersection point y_i of $\bar{Y}_0$ with $\bar{Y}_i$. Here h_i is the conductor of of $\bar{f}_i$ at y_i and m_i is the order of the prime-to-p ramification of $\bar{f}_i$ at y_i. The ramification invariant is the jump in the higher ramification groups of y_i in the upper numbering. Therefore the condition $\sigma_i \neq p/(p-1)$ is automatically satisfied in the geometric setting, since $\gcd(h_i, p) = 1$. We include this here so that Definition 2.3.1 also makes sense for abstract deformation data as in Definition 2.1.1.

The notion of a special deformation datum in [49] corresponds to the case that $r + 1 = 3$. In this case, every G-Galois cover with bad reduction has special reduction. If $r \geq 3$ this is not the case, although this holds "generically" (in some suitable sense). In case $G \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/m$ and $r = 3$ (resp. $G \simeq \mathbb{Z}/p$), these statements are made precise in [8] (resp. [30, Proposition 4.1.1]).

Lemma 2.2.5 implies that $\bar{Z}_0$ is a connected component of the projective nonsingular curve corresponding to the equation

$$z^{p-1} = \prod_{i \in \mathbb{B}'} (x - \tau_i)^{a_i}.$$

Therefore the definition of the a_i and the assumption that the deformation datum is special imply that

$$\omega = \epsilon \bar{\omega}_0, \qquad \text{with} \qquad \bar{\omega}_0 = \frac{z \, \mathrm{d}x}{\prod_{i \in \mathbb{B}'_{\mathrm{prim}}} (x - \tau_i)},$$

for some $\epsilon \in k$.

Put $d_{\mathrm{new}} := |\mathbb{B}_{\mathrm{new}}|$. Lemma 2.1.3 implies that

$$\dim_k H^0(\bar{Z}_0, \Omega)_{\chi_0} = (r + d_{\mathrm{new}}) - \frac{1}{p-1} \sum_{i \in \mathbb{B}} a_i, \qquad \dim_k H^1(\bar{Z}_0, \mathcal{O})_{\chi_0} = (\frac{1}{p-1} \sum_{i \in \mathbb{B}} a_i) - 1. \quad (20)$$

Lemma 2.2.4 together with the assumption that the deformation datum is special implies that

$$\frac{1}{p-1} \sum_{i \in \mathbb{B}} a_i = r + d_{\mathrm{new}} - 1 - \sum_i \nu_i = r - 1. \quad (21)$$

Lemma 2.3.2 *Suppose that $\bar{\omega}$ is a logarithmic differential. Then there exists an element $\Phi_* \in k^\times$ such that*

$$\mathcal{C} \bar{\omega}_0 = \Phi_*^{1/p} \bar{\omega}_0.$$

Proof: We start by computing Φ_*. Write

$$Q = \prod_{i \in \mathbb{B}'_{\mathrm{prim}}} (x - \tau_i)^{1 + a_i} \text{ and } u = \prod_{i \in \mathbb{B}_{\mathrm{new}}} (x - \tau_i)^{a_i}.$$

Then

$$\bar{\omega}_0 = \left(\frac{z}{\prod_{i \in \mathbb{B}'} (x - \tau_i)} \right)^p G \, \frac{\mathrm{d}x}{x},$$

where

$$G = x \frac{\prod_{i \in \mathbb{B}'} (x - \tau_i)^p}{Qu} = \sum_{N=1}^{(d_{\mathrm{new}}+1)p + a_0} g_N x^N.$$

It follows that

$$\mathcal{C} \bar{\omega}_0 = \frac{z}{\prod_{i \in \mathbb{B}'_0} (x - \tau_i)} \frac{\tilde{G} \, \mathrm{d}x}{\prod_{i \in \mathbb{B}_{\mathrm{new}}} (x - \tau_i)}, \qquad \text{with } \tilde{G} = \sum_{N=1}^{d_{\mathrm{new}}+1} g_{pN}^{1/p} x^{N-1}.$$

By assumption, we have that $\omega = \epsilon\bar{\omega}_0$ and $\mathcal{C}\omega = \omega$. Therefore $\mathcal{C}\bar{\omega}_0 = \epsilon^{(p-1)/p}\bar{\omega}_0$. Recall that $\bar{\omega}_0$ has logarithmic poles in the wild critical points τ_i for $i \in \mathbb{B}_{\rm wild} \subset \mathbb{B}_{\rm prim}$, and is holomorphic elsewhere. Therefore $\mathcal{C}\bar{\omega}_0$ is holomorphic outside the wild critical points as well. Therefore $\tilde{G}$ is divisible by $\prod_{i\in\mathbb{B}_{\rm new}}(x-\tau_i)$. Comparing degrees, we conclude that $\tilde{G} = \Phi_*^{1/p}\prod_{i\in\mathbb{B}_{\rm new}}(x-\tau_i)$. Moreover, $\Phi_* = g_{p(d_{\rm new}+1)}$. □

It is easy to see that Φ_* is an expansion coefficient of ω_0 (compare to (6)). If ω is exact, we have that $\mathcal{C}\bar{\omega}_0 = 0$. In analogy with Section 1.2, we call Φ_* the *Hasse invariant*.

Denote by D the divisor $\sum_{i\in\mathbb{B}_{\rm wild}}\tau_i$ on $\bar{Z}_0$. Let $g_R : Z_R \to X_R$ be the H_0-Galois cover associated to the auxiliary cover, as in Section 2.2. Since $f : Y \to \mathbb{P}^1$ has special reduction, it follows that the special fiber $\bar{Z}$ of Z_R is isomorphic to $\bar{Z}_0$. The differential form $\bar{\omega}_0$ lives in $H^0(\bar{Z}_0, \Omega^{\rm log})_\chi \subset H^1_{\rm dR}(\bar{Z}_0/k(\log D))_\chi$ which is isomorphic to $H^1_{\rm dR}(Z_R/R(\log D_R))_\chi \otimes \mathbb{F}_p$, by the above. Here $\Omega^{\rm log} = \Omega^1(\log D)$ and D_R is the lift of the divisor D to a divisor on Z_R induced by $f_R : Y_R \to \mathbb{P}^1_R$. (Recall that $\mathbb{B}_{\rm wild} \subset \mathbb{B}_{\rm prim}$, therefore τ_i for $i \in \mathbb{B}_{\rm wild}$ lifts to a branch point of $f : Y \to \mathbb{P}^1$.) We have that

$$H^1_{\rm dR}(\bar{Z}_0/k(\log D))_\chi / H^0(\bar{Z}_0, \Omega^{\rm log})_\chi \simeq H^1(\bar{Z}_0, \mathcal{O}(-D))_\chi \simeq H^0(\bar{Z}_0, \Omega^{\rm log})^*_\chi.$$

As in Lemma 2.1.3 one checks that $\dim_k H^1(\bar{Z}_0, \mathcal{O}(-D))_\chi = (p-1)(\sum_{i\in\mathbb{B}} a_i) - 1 = r - 2$, by (21).

Now assume that $r = 3$. (This is what we will mostly assume in the rest of this paper.) Then $\dim_k H^1(\bar{Z}_0, \mathcal{O}(-D))_\chi = 1$. Similar to Section 1.1 one show that the element $\bar{\xi} := z\prod_{i\in\mathbb{B}'_{\rm wild}}(x-\tau_i)/x$ forms a basis of $H^1(\bar{Z}_0, \mathcal{O}(-D))_\chi$ in Čech cohomology. Write

$$F\bar{\xi} = \Phi\bar{\xi}.$$

We claim that $\Phi \in k[\lambda]$ is the coefficient of x^{p-1} in $\prod_{i\in\mathbb{B}'}(x-\tau_i)^{a_i}$. This is seen, for example, by noting that

$$\omega^* = \frac{{\rm d}x}{z\prod_{i\in\mathbb{B}'_{\rm wild}}(x-\tau_i)} \in H^0(\bar{Z}_0, \Omega^{\rm log})_{\chi^*}$$

is the dual basis vector to $\bar{\xi}$ under the Serre duality (up to multiplication by an element of $\mathbb{F}_p^\times$). It follows from the properties of the Cartier operator that $\mathcal{C}\omega^* = \Phi^{1/p}\omega^*$, where Φ is as stated above. We call Φ the *dual Hasse invariant*. In Section 4.4 we will discuss the relation between the polynomials Φ and Φ_*. We will mostly consider the case $\mathbb{B}_{\rm wild} = \emptyset$. In this case $\bar{\omega}_0$ is holomorphic, and one may omit the logarithmic poles in the above discussion.

Proposition 2.3.3 *Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover branched at $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$ of order prime to p. Suppose that $(\mathbb{P}^1; x_i)$ is generic, and that f has special reduction to characteristic p. Then the Hasse invariant Φ_* is nonzero.*

Before proving Proposition 2.3.3, we need a some preparation. Let $f : Y \to X = \mathbb{P}^1_K$ be as in the statement of the proposition. Since we suppose that $(X; x_i)$ is generic, we may degenerate f in characteristic zero. For simplicity, we consider the reduction in case $x_3 = \lambda$ goes to $x_1 = 0$. The cover f has so called admissible reduction, which is well understood. The reduction $f_{\rm adm} : Y_{\rm adm} \to X_{\rm adm}$ may be described as follows.

The curve (X, x_i) degenerates to a stably marked curve $(X^{\rm adm}; x_i)$ of genus zero consisting of two irreducible components meeting in one point μ. Denote these two components by X' and X'', where the branch points x_1, x_3 (resp. x_0, x_2) specialize to $X' - \{\mu\}$ (resp. $X'' - \{\mu\}$). We choose a point ρ of $Y^{\rm adm}$ above μ and denote by Y' (resp. Y'') the irreducible component of $Y^{\rm adm}$ above X' (resp. X'') passing through ρ. We denote by $f' : Y' \to X'$ (resp. $f'' : Y'' \to X''$) the covers obtained by restricting $f^{\rm adm}$ to Y' (resp. Y''). These are covers of $\mathbb{P}^1_K$ branched at three points. The fact that $f^{\rm adm}$ is admissible means the following. Let g be the *canonical generator of inertia* at $\rho \in Y'$, with respect to some fixed compatible system of roots of unity in the algebraic closure $\bar{K}$ of K. Then the canonical generator of inertia of $y \in Y''$ is g^{-1}.

Lemma 2.3.4 *At least one of the covers $f' : Y' \to X'$ and $f'' : Y'' \to X''$ has bad reduction.*

Proof: This follows from the assumption that $f : Y \to \mathbb{P}^1_K$ has bad reduction implies that at least one of the covers f' and f'' has bad reduction to characteristic p ([12, Proposition 1.1.4]). □

It is no restriction to suppose that $f'' : Y'' \to X'' \simeq \mathbb{P}^1_K$ has bad reduction. (If not, rename the ramification points x_0, x_1, x_2, x_3.) Let $G'' \subset G$ be the decomposition group of Y''. The cover f'' is branched at at most three points. Recall that we assumed that f'' has bad reduction and that p does not divide the ramification indices of the specializations of x_0 and x_2 to X''. Since x_0 and x_2 specialize to distinct points, this implies that f'' is branched at exactly three points. Write $(\bar{g}''_0 : \bar{Z}''_0 \to \bar{X}''_0, \omega'')$ for the deformation datum induced by the reduction of f''. The results of [50] imply that f'' has special multiplicative reduction. In particular, ω'' is a logarithmic differential form on $\bar{Z}''_0$.

Proof of Proposition 2.3.3: Write $(\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0, \omega)$ for the deformation datum corresponding to $f : Y \to \mathbb{P}^1_K$. Recall that $\bar{Z}_0$ is a connected component of the smooth projective curve given by the Kummer equation

$$z^{p-1} = z^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3} \prod_{i \in \mathbb{B}_{\mathrm{new}}} (x-\tau_i)^{a_i}.$$

The differential form ω is a multiple of

$$\omega_0 := \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)}.$$

If we let $\tau_3 = \lambda$ go to $x_1 = 0$, the cover $\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0$ has admissible reduction, since p does not divide the order of its Galois group. Write $\bar{g}_0^{\mathrm{adm}} : \bar{Z}_0^{\mathrm{adm}} \to \bar{X}_0^{\mathrm{adm}}$ for the corresponding admissible cover of stably marked curves. We denote by $\bar{X}''_0$ (resp. $\bar{X}'_0$) the unique irreducible component of $\bar{X}_0^{\mathrm{adm}}$ such that x_2 and x_0 (resp. x_1 and x_3) specialize to distinct points which are smooth in $\bar{X}_0^{\mathrm{adm}}$. We let $\bar{\mu}$ be the unique point of $\bar{X}''_0$ which is singular in $\bar{X}_0^{\mathrm{adm}}$ and lies in the direction of $\bar{X}'_0$. (This means that the unique path in dual graph of $\bar{X}_0^{\mathrm{adm}}$ which connects the vertices corresponding to $\bar{X}'_0$ and $\bar{X}''_0$ passes through the edge corresponding to $\bar{\mu}$. This is well defined since the dual graph of $\bar{X}_0^{\mathrm{adm}}$ is a tree.)

Write

$$v = \prod_{i \in \mathbb{B}_{\mathrm{new}}} (x-\tau_i)^{a_i}, \qquad d = \deg_x(v).$$

Substituting $\lambda = 0$, we may write

$$v(\lambda = 0) = x^{\delta} w^{p-1} V'',$$

where the order of the zeros of $v''(x)$ is strictly less than $p-1$ and v'' does not have a zero at $x = 0$. Put $d'' = \deg_x(v'')$. It follows that the cover $\bar{g}''_0 : \bar{Z}''_0 \to \bar{X}''_0$ is given by

$$z^{p-1} = x^{a_1+a_3+\delta}(x-1)^{a_2} v'' w^{p-1}.$$

Here we use that $\bar{X}''_0$ occurs in the stable reduction of the cover $f'' : Y'' \to X''$. Define $0 \leq a_\mu < p-1$ by the equivalence $a_\mu \equiv a_1 + a_3 + \delta$. Then $a_0 + a_2 + a_\mu + d'' \equiv 0 \bmod p-1$. Since f'' is a thee-point cover which has special reduction, the vanishing cycle formula (Lemma 2.2.4) implies that $a_0 + a_2 + d'' < p-1$, therefore $a_0 + a_2 + d'' + a_\mu = p-1$.

Write $\tilde{z} = z/(x^{a_1+a_3+\delta-a_\mu} w)$ and

$$\omega''_0 = \frac{\tilde{z}\,\mathrm{d}x}{x(x-1)}.$$

By comparing the orders of zeros and poles, it follows that there exists a $\Phi''_* \in \bar{\mathbb{F}}_p$ such that

$$\omega'' = (\Phi''_*)^{1/(p-1)}\omega''_0.$$

In other words, $\mathcal{C}\omega''_0 = (\Phi''_*)^{1/p}\omega''_0$. We already showed that ω'' is a logarithmic differential, therefore Φ''_* is nonzero.

The explicit expression for ω''_0 shows that ω_0 specializes to ω''_0 on $\bar{Z}''_0$. Since the Hasse invariant satisfies $\mathcal{C}\tilde{\omega}_0 = \tilde{\Phi}_*^{1/p}\tilde{\omega}_0$, it follows that Φ_* specializes to Φ''_*. Therefore Φ_* is nonzero. □

Unfortunately, the proof of Proposition 2.3.3 does not imply that $\Phi \neq 0$, as well. In what follows we will have to impose this as a condition. In Section 4.2.1 we give a sufficient condition for the dual Hasse invariant Φ to be nonzero.

2.4 A lifting lemma In this section we prove a lifting lemma for the auxiliary cover. Our proof essentially follows Wewers [49, Section 3], but our assumptions are not the same as in that paper. The following notations and assumptions replace those of Section 2.3.

The following assumptions and notations replace the previous ones in this section. Let H_0 be a cyclic group of order prime to p and $\chi : H_0 \to \mathbb{F}_p^\times$ a character. Put $G_0 = \mathbb{Z}/p \rtimes_\chi H_0$. Let $(\bar{g}_0, \bar{\omega}_0)$ be a multiplicative deformation datum of type (H_0, χ) (Definition 2.1.1) with $\mathbb{B}_{\rm wild} = \emptyset$. We suppose that the deformation datum is special (Definition 2.3.1). The assumption $\mathbb{B}_{\rm wild} = \emptyset$ is not really necessary. To get rid of it, one should adapt the arguments of [49] as in [50]. We impose the condition to simplify the exposition and since this is the only situation we will use later on in the paper.

Let (σ_i) be the signature of the deformation datum $(\bar{g}_0, \bar{\omega}_0)$ and (τ_i) the set of critical points. It is no restriction to assume that $\tau_0 = \infty, \tau_1 = 0, \tau_2 = 1 \in \mathbb{B}_{\rm prim}$. We write $\mathbb{B}_{\rm prim} = \{0, 1, 2, \ldots, r\}$ and suppose that $\tau_3, \ldots, \tau_r$ define a purely transcendental extension of $\mathbb{F}_p$ of transcendence degree $r-2$.

Let $k = \bar{k}$ be an algebraically closed field of characteristic $p > 0$ such that $(\bar{X}_0; \tau_i)$ are defined over k. In other words, k is the algebraic closure of $\mathbb{F}_p((\tau_i)_{i \in \mathbb{B}'})$. Note that the deformation datum $(\bar{g}_0, \bar{\omega}_0)$ may be defined over k. Let K_0 be the fraction field of $R_0 := W(k)$. Choose an algebraic closure $\bar{K}$ of K_0. For $i \in \mathbb{B}_{\rm prim}$, we choose a lift $x_i \in \mathbb{P}^1(K_0)$ of τ_i such that $\mathbb{Q}_p((x_i)_{i \in \mathbb{B}_{\rm prim}})$ is a purely transcendental extension of $\mathbb{Q}_p$ of transcendence degree $r-2$. It is no restriction to suppose that $x_0 = \infty, x_1 = 0, x_2 = 1$. For every $i \in \mathbb{B}_{\rm new}$, we choose a K_0-rational point $x_i \in \mathbb{P}^1(K_0)$ which lifts $\tau_i \in \mathbb{P}^1(k)$. The goal of this section is to prove the following proposition.

Proposition 2.4.1 *Let $(\bar{g}_0, \bar{\omega}_0)$ be a multiplicative deformation datum of type (H_0, χ) with $\mathbb{B}_{\rm wild} = \emptyset$. There exists a G_0-Galois cover $f_L : Y_L \to \mathbb{P}^1_L$, defined over some field L of characteristic zero, which has bad reduction and whose deformation datum is $(\bar{g}_0, \bar{\omega}_0)$.*

Proof: The H_0-Galois cover $\bar{g}_0$ lifts uniquely to an H_0-Galois cover $g'_{R_0} : Z'_{R_0} \to \mathbb{P}^1_{R_0}$ of smooth curves which is branched along $(x_i) \in \mathbb{P}^1_{R_0}$. We write $g_{K_0} : Z_{K_0} \to \mathbb{P}^1_{K_0}$ for its generic fiber. We write $J(Z_{K_0})$ for the Jacobian variety of Z_{K_0}, and let J_{R_0} be its Néron model. It follows from [5, Section 9, Proposition 4] that J_{R_0} represents the functor $\mathrm{Pic}^0(Z'_{R_0}/R_0)$, since Z'_{R_0} is smooth over R_0. Therefore we have a specialization morphism $J(Z_{K_0})(K_0) \to J(\bar{Z}_0)(k)$, which is surjective by the universal property of the Néron model. In particular, we obtain a surjective morphism

$$J(Z_{K_0})[p](\bar{K})_\chi \to J(\bar{Z}_0)[p](\bar{k})_\chi \tag{22}$$

of $\mathbb{F}_p$-modules.

The (holomorphic) differential form $\bar{\omega}_0$ on $\bar{Z}_0$ is logarithmic, by assumption. Therefore $\bar{\omega}_0$ corresponds to a line bundle $\bar{\mathcal{L}}$ on $\bar{Z}_0$ with the property that $\bar{\mathcal{L}}^{\otimes p} \simeq \mathcal{O}_{\bar{Z}_0}$ ([34, Section III.4]). Concretely, $\bar{\mathcal{L}}$ is defined as follows. Since $\bar{\omega}_0$ is logarithmic, we may write $\bar{\omega}_0 = \mathrm{d}\bar{v}/\bar{v}$, for some rational function $\bar{v}$ on $\bar{Z}_0$. Since $\bar{\omega}_0$ is holomorphic, there exists a divisor D of degree 0 on $\bar{Z}_0$

such that $(\bar{v}) = p \cdot D$. Put $\bar{\mathcal{L}} = \mathcal{O}_{\bar{Z}_0}(D)$, then clearly $\mathcal{L}^{\otimes p} \simeq \mathcal{O}_{\bar{Z}_0}$. It is easy to check that $h^*\bar{\mathcal{L}} = \chi(h)\bar{\mathcal{L}}$, for all $h \in H_0$. Therefore $\bar{\mathcal{L}} \in J(\bar{Z}_0)[p](\bar{k})_\chi$. Since (22) is surjective, we may lift $\bar{\mathcal{L}}$ to $\mathcal{L} \in J(Z_{K_0})[p](\bar{K})_\chi$. By Kummer theory, $\mathcal{L}$ corresponds to a $\boldsymbol{\mu}_p$-torsor $Y_{\bar{K}} \to Z_{\bar{K}}$. After choosing a pth root of unity, we may regard $Y_{\bar{K}} \to Z_{\bar{K}}$ as an étale p-cyclic cover. Let K/K_0 be the minimal extension over which $Y_{\bar{K}} \to Z_{\bar{K}}$ may be defined as Galois cover.

The composition $f_{\bar{K}} : Y_{\bar{K}} \to Z_{\bar{K}} \to \mathbb{P}^1$ is a Galois cover with Galois group $G_0 = \mathbb{Z}_p \rtimes_\chi H_0$. Let L/K_0 be the finite extension over which the stable reduction of $f_{\bar{K}}$ is defined. Since p is totally ramified in L/K_0, the field L is complete; we write R for its ring of integers and ν_R for the corresponding discrete valuation. We write $f_L : Y_L \to \mathbb{P}^1_L$ for the model of $f_{\bar{K}}$ over K. Proposition 2.4.1 is now a consequence of the Lemma 2.4.2. □

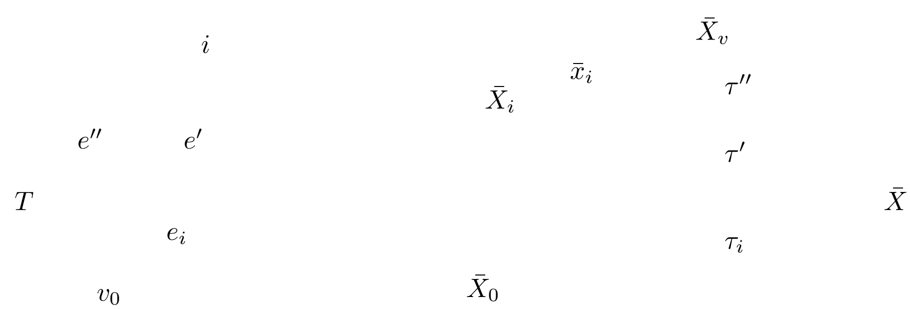
Contrary to the situation of three point covers in [49] it is not true that L/K_0 is a tame Galois cover. We will show in Section 5.2 that for covers branched at 4 points, p strictly divides the degree of L/K_0. However, if $K \subset L$ is a minimal field of definition of f then p does not divide the order of $\mathrm{Gal}(L, K)$. This this the only thing we need in the proof of Lemma 2.4.2.

Lemma 2.4.2 *Write $\bar{f} : \bar{Y} \to \bar{X}$ for the stable reduction of f_L. The deformation datum corresponding to $\bar{f}$ is $(\bar{g}_0, \bar{\omega}_0)$. The curve $\bar{X}$ is a comb. For every $i \in \mathbb{B}$ there is a unique tail of $\bar{X}$; it intersects $\bar{X}_0$ in τ_i.*

Proof: This lemma and its proof are adapted from [49, Proposition 3.5]. It follows from the construction of f_L that f_L has multiplicative bad reduction. Moreover, it is clear that the restriction of $\bar{f}$ to the original component $\bar{X}_0$ of $\bar{X}$ gives rise to the deformation datum $(\bar{g}_0, \bar{\omega}_0)$. Let $\bar{Y}_0$ be the unique irreducible component of $\bar{Y}$ above $\bar{X}_0$. Then $\bar{Y}_0 \to \bar{X}_0$ factors through $\bar{g}_0$ and $\bar{Y}_0 \to \bar{Z}_0$ is a $\boldsymbol{\mu}_p$-torsor.

It remains to show the second part of the lemma. We write $\mathbb{B}^*$ for the set of tails of $\bar{X}$. We want to show that we may identify $\mathbb{B}^*$ with $\mathbb{B}$.

Suppose that $\mathbb{B}^* \neq \mathbb{B}$. It follows from [49, Lemma 3.3] that $\bar{X}_0$ intersects the rest of $\bar{X}$ exactly in the points τ_i. For every $i \in \mathbb{B}$, we denote by T_i the subtree of the dual graph $\bar{X}$ whose root is the edge corresponding to τ_i. The assumption $\mathbb{B}^* \neq \mathbb{B}$ implies that there exists an $i \in \mathbb{B}$ such that T_i contains more than one tail. It is proved in [49, Lemma 3.3] that this may only happen for $i \in \mathbb{B}_{\mathrm{new}}$ and that in this case T_i contains two tails. More precisely, there exists a component $\bar{X}_v$ of $\bar{X}$ which intersects $\bar{X}_0$ in τ_i and two tails which intersect $\bar{X}_v$. Write τ', τ'' for the intersection points of $\bar{X}_v$ with the two tails. We may suppose that x_i specializes to the tail $\bar{X}_i$ which intersects $\bar{X}_v$ in τ'', as illustrated by the following picture. Let $\bar{Y}_v$ be a component of $\bar{Y}$ above $\bar{X}_v$ and let $\bar{Z}_v$ be the corresponding component of $\bar{Z}$. Then $\bar{Z}_v \to \bar{X}_v$ is totally branched at τ'' and τ_i.



Let K be as in the proof of Proposition 2.4.1. Recall that K contains a pth-root of unity. Let $\Gamma = \mathrm{Gal}(L, K)$. It follows that Γ acts faithfully on $\bar{Y}$. As in the proof of [49, Proposition 3.5] and [40, Section 4.2] it follows that Γ acts trivially on $\bar{Y}_0$. In particular, if we choose a point $\bar{y}$ of $\bar{Y}$ above τ_i, then $\bar{y}$ is fixed by Γ.

Write $\hat{\Gamma}$ for the image of Γ in the group of automorphism of $\bar{Y}_v$. As in the proof of [49, Proposition 3.5], it follows that $\hat{\Gamma}$ acts trivially on $\bar{X}_v$. The argument of step 3 of the proof of [40, Proposition 4.2.4] implies that $\hat{\Gamma}$ has order prime to p. Therefore the proof of [49, Proposition 3.5] applies to our case. This proves the lemma. □

3 Existence of special deformation data

In this section we show that a deformation datum (g_k, ω) corresponds to a solution u of a Fuchsian differential equation with certain properties. The signature of the deformation datum corresponds to the local exponents of the corresponding Fuchsian differential equation (i.e. the restricted Riemann scheme). The additional properties we require for u is a residue condition on the Wronskian of the differential equation at the new critical points (Proposition 3.1.1.(b)). This description translates the existence of a deformation datum with given signature into the existence of an algebraic solution of a Fuchsian differential equation with given restricted Riemann scheme. This last problem is Dwork's accessary parameter problem. Since the solution u is required to have additional properties, we are lead to study a variant of the accessary parameter problem.

The main results of this section are the following. If the number of accessary parameters is one, we show that there always exists a deformation datum with given signature $\boldsymbol{\sigma}$ (Proposition 3.3.2). In the general case, we show that the deformation functor of deformation data with signature $\boldsymbol{\sigma}$ is formally smooth and one dimensional (Lemma 3.4.2). Therfore is there exists a deformation datum with signature $\boldsymbol{\sigma}$, then there exists such a deformation datum for which the marked curve $(X_k; \tau_0 = \infty, \tau_1 = 0, \tau_2 = 1, \tau_3 = \lambda)$ is generic (Proposition 3.4.3). As an application, we show that the accessary parameter cover $\pi : B_0 \to \mathbb{P}^1_\lambda$ is separable.

3.1 Relation with solutions of Fuchsian differential equations in characteristic p Suppose that we are given a special multiplicative deformation datum (g, ω) of type (H, χ) with $\chi : H \to \mathbb{F}_p^\times$ injective. Recall from Definition 2.3.1 that this means that $\mathbb{B} = \mathbb{B}_{\rm prim} \cup \mathbb{B}_{\rm new}$, where $\mathbb{B}_{\rm prim} = \{i \in \mathbb{B} \,|\, 0 \leq \sigma_i \leq 1\}$ and $\mathbb{B}_{\rm new} = \{i \in \mathbb{B} \,|\, 1 < \sigma_i \leq 2\}$. The goal of this section is to associate to (g, ω) a polynomial solution of a Fuchsian differential equation in characteristic p. Similar results occur in [10] and [13]. For the results of this section, it is not necessary to assume that the deformation datum is special and multiplicative, see [10]. However, this is the only case we will be interested in in what follows.

Put $r + 1 = |\mathbb{B}_{\rm prim}|$. choose a subset $\mathbb{B}_{\rm ns}$ of $\{i \in \mathbb{B}_{\rm new} \,|\, \sigma_i = (p+1)/(p-1)\}$. We call $\mathbb{B}_{\rm ns}$ the set of *nonsingular critical points*. The reason for this terminology is the following. To our family of covers $Z_k \to \mathbb{P}^1_k$, we will associate a Fuchsian differential equation which does not have singularities in the nonsingular new tails.

Write $\mathbb{B}_0 = \mathbb{B} - \mathbb{B}_{\rm ns}$ and put $s + 1 = |\mathbb{B}_0|$. We may assume that the critical points τ_i for $i \in \mathbb{B}_{\rm prim}$ contain $0, 1, \infty$. We suppose that $\tau_0 = \infty$. Then $\tau_i \neq \infty$ for $i \in \mathbb{B}_{\rm new}$. Write $\mathbb{B}'_{\rm prim} = \{i \in \mathbb{B}_{\rm prim} \,|\, \tau_i \neq \infty\}$ and $\mathbb{B}'_0 = \{i \in \mathbb{B}_0 \,|\, \tau_i \neq \infty\}$. It is no restriction to suppose that $\mathbb{B}_{\rm prim} = \{0, 1, \ldots, r\}$ and $\mathbb{B}_0 = \{0, 1, \ldots, s\}$.

Suppose that $\mathbb{B}_{\rm ns} \neq \emptyset$. Define $u(x) = \prod_{i \in \mathbb{B}_{\rm ns}} (x - \tau_i)$, and let d be the degree of u. It follows that the curve Z_k is a connected component of the smooth projective curve defined by

$$z^{p-1} = \prod_{i \in \mathbb{B}'_0} (x - \tau_i)^{a_i} u^2.$$

Here we use that $a_i = 2$ for $i \in \mathbb{B}_{\rm ns}$. Moreover, it follows that

$$\omega = \frac{\epsilon z \, {\rm d}x}{\prod_{i \in \mathbb{B}'_{\rm prim}} (x - \tau_i)},$$

for some $\epsilon \in k^\times$.

The Riemann–Roch formula, together with the assumptions we made on the signature, implies that

$$2d + \sum_{i \in \mathbb{B}_0} a_i = (p-1)(r-2)$$

(Lemma 2.2.4.(a)). In particular, this implies that $\sum_{i\in\mathbb{B}_0} a_i$ is even.

Write

$$Q := \prod_{i\in\mathbb{B}_0'} (x-\tau_i)^{1+a_i-\nu_i}, \quad P_0 := \prod_{i\in\mathbb{B}_0'} (x-\tau_i), \quad P_1 := Q'P_0/Q.$$

Put $\gamma_1 = -d$ and $\gamma_2 = \gamma_1 - a_0$. Recall that a_0 corresponds to $\tau_0 = \infty$.

In the following proposition, we show that for fixed $(\tau_i)_{i\in\mathbb{B}_0}$, the nonsingular critical points $(\tau_i)_{i\in\mathbb{B}_{\rm ns}}$ are characterized by the fact that u is a solution to a Fuchsian differential equation.

Proposition 3.1.1 *Let (g,ω) be a special, multiplicative deformation datum of signature (σ_i). Suppose that $d = |\mathbb{B}_{\rm ns}| > 0$. Let κ be the algebraic closure of the field obtained by adjoining τ_i for $i \in \mathbb{B}_0'$ to k.*

(a) There exists a polynomial $P_2 = \sum_{j=0}^{s-2} \beta_j x^j$ with $\beta_j \in \kappa$ and $c_{s-2} = \gamma_1 \cdot \gamma_2$ such that $u := \prod_{i\in\mathbb{B}_{\rm new}}(x-\tau_i)$ is the solution to the Fuchsian differential equation

$$P_0 u'' + P_1 u' + P_2 u = 0. \tag{23}$$

(b) The function

$$w = \frac{1}{Qu^2}$$

satisfies

$$\mathrm{Res}_{\tau_i} w = 0, \qquad \text{for } i \in \mathbb{B}_{\rm new}.$$

The coefficients $\beta_0, \ldots, \beta_{s-3}$ of P_2 are called the *accessary parameters* of the differential equation. The number of accessary parameters is $s-2$. The differential equation (23) has singularities at $x = \tau_i$ for $i \in \mathbb{B}_0$. The local exponents at τ_i are $0, -a_i+\nu_i$ for $i \in \mathbb{B}_0'$ and γ_1, γ_2 at ∞.

Proof: The following proof is inspired by [4, Lemma 3]. A similar argument can be found in [19, Theorem 5]. See also [13, Proposition 3.2] and [10, Lemma 4.2]. Write

$$F = \epsilon \frac{z}{\prod_{j\in\mathbb{B}_{\rm prim}'}(x-\tau_j)} = \frac{\epsilon z^p}{Qu^2}.$$

We suppose that $\omega = F\,\mathrm{d}x$ is logarithmic. This is equivalent to $D^{p-1}F = -F^p$, where $D := \partial d/\partial dx$. Since $D^{p-1}F = \epsilon z^p D^{p-1}[1/(Qu^2)]$, we find

$$D^{p-1}\frac{1}{Qu^2} = -\epsilon^{p-1}\frac{1}{\prod_{i\in\mathbb{B}_{\rm prim}'}(x-\tau_i)^p}. \tag{24}$$

Choose $i \in \mathbb{B}_{\rm new}$ and write

$$\frac{1}{Qu^2} = \sum_{n\geq -2} c_n (x-\tau_i)^n.$$

Then

$$D^{p-1}\frac{1}{Qu^2} = -[\frac{c_{-1}}{(x-\tau_i)^p} + c_{p-1} + \cdots] = -\epsilon^{p-1}\frac{1}{\prod_{i\in\mathbb{B}_{\rm prim}'}(x-\tau_i)^p}.$$

We conclude that $c_{-1} = 0$, since $i \notin \mathbb{B}_{\text{prim}}$.

Write

$$
\begin{aligned}
Qu^2 &= [Q(\tau_i) + Q'(\tau_i)(x-\tau_i) + \cdots][u'(\tau_i)(x-\tau_i) + \frac{1}{2}u''(\tau_i)(x-\tau_i)^2 + \cdots]^2 \qquad (25)\\
&= (x-\tau_i)^2\,[\,Q(\tau_i)u'(\tau_i)^2 + u'(\tau_i)(\,Q'(\tau_i)u'(\tau_i) + Q(\tau_i)u''(\tau_i)\,)\,(x-\tau_i) + \cdots]. \qquad (26)
\end{aligned}
$$

We see that $-c_{-1} = u'(\tau_i)[\,Q'(\tau_i)u'(\tau_i) + Q(\tau_i)u''(\tau_i)\,]$. Since $u'(\tau_i) \neq 0$, we have that $Q'(\tau_i)u'(\tau_i) + Q(\tau_i)u''(\tau_i) = 0$ for all $i \in \mathbb{B}_{\text{new}}$.

Define $G = Q'u' + Qu''$. This is a polynomial of degree less than or equal to $e := \deg(Q) + \deg(u) - 2$. The coefficient of x^e in G is $g_e := \deg(u)(\deg(Q) + \deg(u) - 1) = -\gamma_1\gamma_2$. The polynomial G is divisible by $u\prod_{i\in\mathbb{B}_0'}(x-\tau_i)^{a_i}$. Therefore $G = \prod_{i\in\mathbb{B}_0'}(x-\tau_i)^{a_i}uR$, where R is a polynomial of degree less than or equal to $s-2$. The coefficient of R of degree $s-2$ is g_e. Dividing by $\prod_{i\in\mathbb{B}_0'}(x-\tau_i)$, we find that u is a solution to the Fuchsian differential equation

$$P_0u'' + P_1u' + P_2u = 0.$$

This proves (a).

We have already shown that the residue of w at $x = \tau_i$ is zero for $i \in \mathbb{B}_{\text{ns}}$. For $i \in \mathbb{B}_{\text{new}} \cap \mathbb{B}_0$, this follows from (24). □

3.2 A converse to Proposition 3.1.1 In this section we prove a converse to Proposition 3.1.1. The result is a generalization of [10, Proposition 4.3]. We start by recalling the notation.

Let $s+1 \geq r+1 \geq 3$ be integers. Suppose given $0 \leq a_0, a_1, \ldots, a_s < p-1$ such that $\mathcal{A} := a_0 + a_1 + \cdots + a_s$ is an even integer less that $(r-1)(p-1)$, and let $d = ((r-2)(p-1) - \mathcal{A})/2$. We assume moreover that $a_{r+1}, \ldots, a_s$ are different from 1. For an explanation of this condition, see Section 2.3. Put $\mathbb{B}_{\text{prim}} = \{0, 1, \ldots r\}$ and $\mathbb{B}_0 = \{0, 1, \ldots, s\}$. Define $\nu_i = 0$ if $i \in \mathbb{B}_{\text{prim}}$ and $\nu_i = 1$ otherwise. For $i \in \mathbb{B}_0$ we choose pairwise distinct points $\tau_i \in \mathbb{P}^1_k$, where we suppose that $\tau_0 = \infty, \tau_1 = 0, \tau_2 = 1$. Define

$$P_0 = \prod_{i=1}^{s}(x-\tau_i), \quad Q = \prod_{i=1}^{s}(x-\tau_i)^{1+a_i-\nu_i}, \quad p_1 = \frac{Q'}{Q}, \quad p_2 = \frac{d(d+a_0)x^{s-2} + \beta_{s-3}x^{s-3} + \cdots + \beta_0}{P_0}$$

and

$$L(u) = u'' + p_1u' + p_2u = 0. \qquad (27)$$

Let κ be an algebraically closed extension of k which contains $\tau_1, \ldots, \tau_s$. Let $\gamma_1 = -d$ and $\gamma_2 = -d - a_0$. Recall that these are the local exponents of $L(u) = 0$ at $x = \infty$.

For future reference, we note the following properties of L.

Lemma 3.2.1 *Let $u = u(x)$ be a polynomial solution of $L(u) = 0$. Then*

(a) $\deg(u) \equiv -\gamma_j \bmod p$, *for* $j = 1, 2$,

(b) $\text{ord}_{\tau_i}(u) \equiv 0$ *or* $\text{ord}_{\tau_i}(u) \equiv -a_i + \nu_i \bmod p$, *for* $i = \mathbb{B}_0'$.

Proof: Let $i \in \mathbb{B}_0$ and choose a local parameter t of $\mathbb{P}^1$ at τ_i. Rewriting the differential equation in terms of t immediately implies the lemma; (a) corresponds to $i = 0$ and (b) corresponds to $i \neq 0$. □

For the definition of *sufficiently many solutions in a weak sense*, we refer to Section 4.1.

Lemma 3.2.2 *Suppose that (27) has a nonzero solution $u \in \kappa[x]$. Then it has sufficiently many solutions in a weak sense.*

Proof: This follows immediately from the definition, as $L_W(w) = w' + p_1 w = 0$ has a solution $w = 1/Q \in \kappa(x)$. □

Proposition 3.2.3 *Suppose that there exist $\beta_0, \ldots, \beta_{s-3} \in \kappa$ such that $L(u) = 0$ has a solution $u \in \kappa[x]$ which satisfies:*

- $\deg(u) = d$,
- $u(\tau_i) \neq 0$ *for* $i \in \mathbb{B}'_0$,
- $\mathrm{Res}_{\tau_i} 1/Qu^2 = 0$ *for* $i \in \mathbb{B}_0 - \mathbb{B}_{\mathrm{prim}}$.

Let $Z_k \to \mathbb{P}^1_k$ be the cyclic cover of smooth curves defined by taking an irreducible component of

$$z^{p-1} = \prod_{i=1}^{s-1} (x - \tau_i)^{a_i} u^2.$$

Then, for suitable $\epsilon \in \kappa^\times$, the differential

$$\omega = \frac{\epsilon z \,\mathrm{d}x}{\prod_{i=1}^{r-1}(x - \tau_i)}$$

on Z_k defines a special deformation datum.

Proof: Let $\beta_0, \ldots, \beta_{s-3}, u, Z_k, \omega$ be as in the statement of the proposition. We have seen that ω is logarithmic if and only if

$$D^{p-1} \frac{1}{Qu^2} = -\epsilon^{p-1} \frac{1}{\prod_{i=1}^{r-1}(x - \tau_i)^p}, \qquad \text{where } D = \partial d/\partial dx.$$

Similarly, ω is exact if and only if

$$D^{p-1} \frac{1}{Qu^2} = 0.$$

Since u is a solution to $L(u) = 0$ which does not have zeros in the set $\mathbb{B}'_0$ of (finite) singularities, it follows that u has at most simple zeros. Therefore we may write

$$u = \prod_{i \in \mathbb{B}_{\mathrm{ns}}} (x - \mu_i),$$

where $(\mu_i)_{i \in \mathbb{B}_{\mathrm{ns}}}$ are pairwise disjoint and different from the τ_i for $i \in \mathbb{B}_0$. As in the proof of Proposition 3.1.1, one checks that the residue of $1/Qu^2$ at μ_i is zero.

Consider the partial fraction decomposition of $1/Qu^2$:

$$\frac{1}{Qu^2} = \sum_{i=1}^{s} \sum_{j=1}^{1+a_i-\nu_i} \frac{\rho_i(j)}{(x - \tau_i)^j} + \sum_{i \in \mathbb{B}_{\mathrm{ns}}} \frac{b_i}{(x - \mu_i)^2}.$$

By assumption, $\rho_i(1) = 0$ for $i \in \mathbb{B}_0 - \mathbb{B}_{\mathrm{prim}}$. Therefore

$$D^{p-1} \frac{1}{Qu^2} = -\sum_{i=1}^{r} \frac{\rho_i(1)}{(x - \tau_i)^p} =: -\frac{d_{r-1} x^{(r-1)p} + d_{r-2} x^{(r-2)p} + \cdots + d_0}{\prod_{i=1}^{r}(x - \tau_i)^p},$$

where

$$d_i = \pm \sum_{1 \le j_1 < \cdots < j_i \le r} \left(\rho_{j_1}(1) \cdots \rho_{j_i}(1) \prod_{\ell \neq j_1, \ldots, j_i} \tau_\ell \right).$$

Claim: $d_{r-1} = \cdots = d_1 = 0$.

Note that the claim implies that there exists an $\epsilon \in \kappa^\times$ such that ω is logarithmic if and only if $d_0 \neq 0$. Moreover, if $d_0 = 0$ then ω is exact, for every choice of ϵ. Therefore the proposition follows from the claim.

To prove the claim we apply the Cartier operator to ω:

$$\mathcal{C}\omega = \epsilon^{1/p} z \left(\sum_{i=1}^{r} \frac{\rho_i(1)^{1/p}}{x - \tau_i} \right) \mathrm{d}x.$$

For a point $z = \infty$ of Z_k above $\infty \in \mathbb{P}^1_k$ we have that

$$\mathrm{ord}_\infty \omega = \frac{a_0}{\gcd(p-1, a_0)} - 1 = \begin{cases} -1 & \text{if } a_0 = 0, \\ \geq 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{C}\omega$ has a simple pole above $x = \infty$ if $a_0 = 0$ and is regular otherwise. One computes that

$$\mathrm{ord}_\infty z\,\mathrm{d}x = -r(p-1) + a_0 - \gcd(a_0, p-1).$$

This implies that $d_i = 0$ for $i = 1, \ldots, r-1$. □

In Section 3.4 we show that, for fixed position of the primitive singularities $\tau_i, i \in \mathbb{B}_{\mathrm{prim}}$, there are at most finitely many solutions u as in Proposition 3.2.3. Something similar, but weaker, is proved in [15]. In case $s+1 = r+1 = 4$, this follows from the results of Section 3.3. If we fix the τ_i for $i \in \mathbb{B}_{\mathrm{prim}}$, the differential equation $L(u) = 0$ has $s - r$ varying branch points and $s - 2$ accessary parameters. To show the existence of u as in Proposition 3.2.3, one has to solve $2s - r$ equations. (This will be illustrated in the next sections.) Alternatively, one could work directly with the equations coming from $\mathcal{C}\omega = \omega$ (resp. $\mathcal{C}\omega = 0$), depending on whether ω should be logarithmic or exact. This is done for example in [11]. In this case one has $s + d - r$ variables, corresponding to the moving singularities and the nonsingular critical points μ_i. This means that if s is small with respect to d, the method of Proposition 3.2.3 is more efficient. In practice, this becomes rather complicated as soon as the number of accessary parameters is larger than one, therefore in concrete examples we mainly restrict to the case $s = r = 3$.

3.3 The existence of special deformation data In this section, we analyze the existence of special deformation data with given signature if $r+1 = s+1 = |\mathbb{B}_{\mathrm{prim}}| = 4$. For analogous results in the case $r = 2$, see [13]. We now show the existence of polynomial solutions of (27), for suitable choice of the accessary parameter $\beta = \beta_0$. Let $0 \leq a_0, \ldots, a_3 < p-1$ be nonnegative integers such that $\mathcal{A} := \sum_{i=0}^{3} a_i$ is even and suppose that $\mathcal{A} < 2(p-1)$. Put $d := (p-1) - \mathcal{A}/2$. We suppose that $\tau_0 = \infty, \tau_1 = 0, \tau_2 = 1, \tau_3 = \lambda$ with λ transcendental. Put $\kappa = k((\lambda))$. Then

$$P_0 = x^3 - (1+\lambda)x^2 + \lambda x, \tag{28}$$

$$P_1 = x^2(3 + a_1 + a_2 + a_3) - x(\lambda(2 + a_1 + a_2) + 2 + a_1 + a_3) + \lambda(1 + a_1), \tag{29}$$

$$P_2 = \gamma_1\gamma_2 x + \beta. \tag{30}$$

Recall that the differential equation $P_0 y'' + P_1 y' + P_2 y = 0$ has local exponents $0, -a_i$ at τ_i for $i = 1, 2, 3$ and $\gamma_1 = -d, \gamma_2 = -d - a_0$ at $\tau_0 = \infty$. We describe the choice of the accessary parameter for which the differential equation has a solution of degree d as an eigenvalue problem, following [4].

Write V_d for the set of polynomials over k of degree less than or equal to d. For $= 1, 2, 3$, we write V_d^i for the subset of V_d of polynomials which have a zero at τ_i of order at least $p - a_i$. Let $\mathbb{L} = P_0(\partial/\partial x)^2 + P_1 \partial/\partial x + \gamma_1\gamma_2 x$.

We claim that the differential operator $\mathbb{L}$ acts on V_d and V_d^i. For every integer j we have

$$\mathbb{L}(x^j) = (j + \gamma_1)(j + \gamma_2)x^{j+1} + (\cdots)x^j + (j + a_1)j\lambda x^{j-1}.$$

In particular $\mathbb{L}(x^j)$ has degree less than or equal to $j+1$. Since $\gamma_1 = -d$, it follows that $\mathbb{L}(x^d)$ has degree $\leq d$. This shows that $\mathbb{L}$ acts on V_d.

To show that $\mathbb{L}$ acts on V_d^i it suffices to consider $i = 1$. Let $u \in V_d^1$. The above computation shows that $\mathbb{L}(u)$ has a zero of order at least $p - a_1$ also. This proves the claim.

A solution $u \in V_d$ of the differential equation satisfies $\mathbb{L}(u) = \beta u$. We write χ_d (resp. χ_d^i) for the characteristic polynomial of $\mathbb{L}$ on V_d (resp. V_d^i).

Lemma 3.3.1 *The dimension of $V_d^1 + V_d^2 + V_d^3$ is strictly less than the dimension of V_d.*

Proof: One checks that $V_d^1 \cap V_d^2 \cap V_d^3 = (0)$, since $a_i < p - 1$. Therefore

$$\dim(V_d^1 + V_d^2 + V_d^3) = \sum_{i=1}^{3} \max(d + 1 - p + a_i, 0) - \sum_{1 \leq i < j \leq 3} \max(d + 1 - 2p + a_i + a_j, 0).$$

If $a_i \leq p - 1 - d = \mathcal{A}/2$ for $i = 1, 2, 3$ then $\dim V_d^i = 0$. In this case the lemma holds.

The equality $a_0 + a_1 + a_2 + a_3 = 2(p - 1 - d)$ implies that there is at most one $i \in \{0, 1, 2, 3\}$ such that $a_i > p - 1 - d$. Suppose that $a_1 > p - 1 - d$. Then $V_d^1 + V_2^d + V_3^d = V_1^d$ has dimension $d + 1 + a_1 - p$, which is strictly less than $\dim V_d = d + 1$. □

The following proposition show the existence of special deformation data with given signature. For simplicity, we assume that $2a_0 \leq \mathcal{A} = 2(p - 1 - d)$. The proof of Lemma 3.3.1 implies that we may always assume this, after renumbering the branch points if necessary.

Proposition 3.3.2 *Suppose that $2a_0 \leq \mathcal{A}$.*

(a) There exists a $\beta \in \kappa$ such that the differential equation

$$P_0 u'' + P_1 u' + P_2(\beta) u = 0$$

has a polynomial solution u of degree d with $u(0) \cdot u(1) \cdot u(\lambda) \neq 0$.

(b) For given λ, the number of β as in (a) is finite.

(c) For given β as in (a), there exists a unique monic solution u of degree d.

Proof: The discussion above Lemma 3.3.1 shows that the differential equation $\mathbb{L} = \mathbb{L}_\beta$ has a solution of degree $\leq d$ if and only if $\chi_d(\beta) = 0$. Since $\chi_d(t) \in k[\lambda][t]$ is a polynomial in t of degree $d + 1$, such β always exists in a finite extension of $k[\lambda]$.

The assumption $2a_0 \leq \mathcal{A}$ implies that $0 < d < d + a_0 < p$. Therefore it follows from Lemma 3.2.1.(a) that if $u \in V_d$ is a solution of the differential equation (for some β) then $\deg(u) = d$.

Lemma 3.3.1 implies that there exists a β such that $\chi_d(\beta) = 0$ but $\chi_d^i(\beta) \neq 0$ for $i = 1, 2, 3$. Choose β like this and let u be the corresponding solution of the differential equation. Since $u \notin V_d^i$, it follows that u does not have a zero in $x = 0, 1, \lambda$. This proves (a). Part (b) is immediate.

To prove (c), we use the following notation. We let β and u be as above. Define

$$\begin{aligned} A_i &= (i+1)(i+1+a_1), \quad B_i = (i + \gamma_1 - 1)(i + \gamma_2 - 1), \\ C_i &= i^2(1+\lambda) + i(\lambda(1 + a_1 + a_2) + 1 + a_1 + a_3). \end{aligned}$$

Since $u = \sum_i u_i x^i$ is a solution of the differential equation $P_0 u'' + P_1 u' + P_2 u = 0$, one checks that the u_i satisfy the recursion

$$\lambda A_i u_{i+1} = (C_i - \beta) u_i - B_i u_{i-1}. \tag{31}$$

If $p - a_1 > d$ then $A_i \neq 0$ for $0 \leq i \leq d$. In this case the recursion immediately implies that the coefficients u_i are uniquely determined by u_0.

Suppose that $p - a_1 \leq d$. Then $A_{p-1-a_1} = 0$ and $A_i \neq 0$ for $0 \leq i \leq d$ with $i \neq p - 1 - a_1$. Therefore u_i is uniquely determined by u_0 and β for $i \leq p - 1 - a_1$. It follows that β satisfies

$$0 = (C_{p-1-a_1} - \beta)u_{p-1-a_1} - B_{p-1-a_1}u_{p-2-a_1}.$$

The values u_i for $p - a_1 < i \leq d$ are uniquely determined by u_0, u_{p-a_1} and β. The value u_{p-a_1} is determined by the condition $0 = (C_d - \beta)u_d - B_d u_{d-1}$. This condition is linear in u_{p-a_1}. Since we know that a solution u exists, it follows that u is uniquely determined by u_0 and β. □

3.4 The accessary parameter problem In this section we discuss a variant of a result of Dwork ([15]) on the accessary parameter problem. Fix a type (σ_i) as before, and let (a_i) be as defined in (17). Recall that giving the (σ_i) is equivalent to giving the local exponents of the differential equation (27). Roughly speaking, Dwork shows that the locus of all (τ_i, β_i) such that the differential equation (27) has nilpotent but nonzero p-curvature is locally a complete intersection which is a finite cover of the space of singularities (τ_i). Similar results where proven by Mochizuki ([35], [36]). We refer to Section 4.1 for the definition of the p-curvature. In our terminology, this implies that for given signature (σ_i) and set of critical points $(\tau_i)_{i\in\mathbb{B}_0}$, the number of accessary parameters (β_i) such that the differential equation (27) admits an algebraic solution u is finite.

The variant of this problem we want to discuss is the following. We fix a special signature (σ_i) together with the set of primitive critical points $(\tau_i)_{i\in\mathbb{B}_{\rm prim}}$. We want to show that the number N of corresponding deformation data is finite. More concretely, N is the number of $(\tau_i)_{i\in\mathbb{B}_0\cap\mathbb{B}_{\rm new}} \cup (\beta_i)$ such that the differential equation (27) has a solution u which satisfies the conditions of Proposition 3.2.3. In case $\mathbb{B}_0 \cap \mathbb{B}_{\rm new} = \emptyset$ and $|\mathbb{B}_{\rm prim}| = r + 1 = 4$ this follows already from the results of Section 3.3.

Since we want to use similar techniques in Section 6.2 to give a criterion for special reduction, we need to consider a somewhat more general set-up than would be needed for the results of this section. Let R be a complete discrete valuation ring with fraction field L of characteristic zero and algebraically closed residue field k of characteristic $p > 0$. Let G be a group whose order is strictly divisible by p. Let $f_L : Y_L \to \mathbb{P}^1_L$ be a G-Galois cover defined over L branched at $r + 1 = 4$ points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$ of order prime to p. We suppose that $(\mathbb{P}^1_L; x_0, x_1, x_2, x_3)$ has good reduction, and $f_L : Y_L \to \mathbb{P}^1_L$ has multiplicative bad reduction $\bar{f} : \bar{Y} \to \bar{X}$. Note that at the moment we do not assume that the reduction is special. Choose an irreducible component $\bar{Y}_0$ of $\bar{Y}$ above the original component $\bar{X}_0$. Let $(\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0, \omega)$ be the corresponding deformation datum. Our assumptions imply that ω is a holomorphic, logarithmic differential form.

Since we do not assume that the deformation datum is special, we need to slightly adapt our notation. It coincides with the usual notation if the deformation datum is special. We let $\mathbb{B}$ be the set of critical points of the deformation datum, and write $\sigma_i = \nu_i + a_i/(p-1)$ with $0 \leq a_i < p - 1$. We denote by $\mathbb{B}_{\rm prim} = \{0, 1, 2, 3\} \subset \mathbb{B}$ the set of primitive critical points and $\mathbb{B}_{\rm new} = \mathbb{B} - \mathbb{B}_{\rm prim}$ the set of new critical points. Let $\mathbb{B}_{\rm ram} = \{i \in \mathbb{B} \,|\, a_i \neq 0\}$.

We write $\bar{f}^{\rm aux} : \bar{Y}^{\rm aux} \to \bar{X}$ for the auxiliary cover, as defined in Section 2.2. See also Section 5.2 for a more detailed discussion. As usual, $(\sigma_i)_{i\in\mathbb{B}}$ (resp. $(\tau_i)_{i\in\mathbb{B}}$) denotes the signature (resp. the set of critical points) of the deformation datum. We choose a set $\mathbb{B}_{\rm ns} \subset \{i \in \mathbb{B}_{\rm new} \,|\, \sigma_i = (p+1)/(p-1)\}$ of nonsingular critical points, and let $\mathbb{B}_0 = \mathbb{B} - \mathbb{B}_{\rm ns}$. Since $\mathbb{B}_{\rm wild} = \emptyset$ and $|\mathbb{B}_{\rm prim}| = 4$, it follows from Lemma 2.2.4 that

$$\frac{1}{p-1}(\sum_{i\in\mathbb{B}} a_i) \in \{1, 2\}.$$

The reduction is special if and only if this sum equals 2.

Let $G_0 \subset G$ be the decomposition group of $\bar{Y}_0$. Recall that $G_0 \simeq I_0 \rtimes_\chi H_0$ where I_0 is a Sylow p-subgroup of G, which has order p, and $\chi : H_0 \to \mathbb{F}_p^\times$ is a nontrivial character. We denote by $\mathcal{G}_0$ the group scheme $\boldsymbol{\mu}_p \rtimes_\chi H_0$, as defined in [51, Section 4.1]. We associate to the deformation datum $(\bar{g}_0, \omega)$ a singular curve $Y_{\rm sing}$ together with an action of the group scheme $\mathcal{G}_0$, as in [51, Construction 4.3]. Since ω is a logarithmic differential, locally on $\bar{Z}_0$ it may be written as

$$\omega = \frac{\mathrm{d}h}{h}.$$

Define $\bar{Y}_{\rm sing}$ locally on $\bar{Z}_0$ by the equation $y^p = h$. Then obviously $\mathcal{G}_0$ acts on $\bar{Y}_{\rm sing}$ and the natural map $\bar{Y}_{\rm sing} \to \bar{X}_0$ is a $\mathcal{G}_0$-torsor outside the branch points of $\bar{Z}_0 \to \bar{X}_0$ ([51, Remark 4.6.i]). Moreover, [51, Remark 4.6.ii] implies that $\bar{Y}_{\rm sing}$ is generically smooth.

Let $\mathfrak{C}_k$ be the category of local artinian k-algebras of equal characteristic p. A *$\mathcal{G}_0$-equivariant deformation* of $\bar{Y}_{\rm sing}$ over an object A of $\mathfrak{C}$ is a flat R-scheme $\bar{Y}_R$ together with an action of $\mathcal{G}_0$ and an $\mathcal{G}_0$-equivariant isomorphism $\bar{Y}_{\rm sing} \simeq \bar{Y}_R \otimes_R k$. We consider the deformation functor

$$R \mapsto \mathrm{Def}\,(\bar{Y}_{\rm sing}, \mathcal{G}_0)(R)$$

which sends $R \in \mathfrak{C}_k$ to the set of isomorphism classes of $\mathcal{G}_0$-equivariant deformations of $\bar{Y}_{\rm sing}$ over R. Let

$$R \mapsto \mathrm{Def}\,(\bar{X}_0; \tau_i \,|\, i \in \mathbb{B}_{\rm ram})(R)$$

be the deformation functor which sends R to the set of isomorphism classes of deformations of the pointed curve $(\bar{X}_0; \tau_i \,|\, i \in \mathbb{B}_{\rm ram})$. We consider the points τ_i on $\bar{X}_0$ to be ordered and up to the action of $\mathrm{PGL}_2(k)$. We obtain a natural transformation

$$\mathrm{Def}\,(\bar{Y}_{\rm sing}, \mathcal{G}_0) \longrightarrow \mathrm{Def}\,(\bar{X}_0; \tau_i \,|\, i \in \mathbb{B}_{\rm ram}). \tag{32}$$

In the situation of [51, Sections 4 and 5] the natural transformation (32) is an isomorphism. This is no longer always the case in our situation as the following lemma shows. This is where we use the assumption that $r + 1 = 4$.

Lemma 3.4.1 *(a) If $\bar{f}$ is special, the natural transformation (32) is a $\boldsymbol{\mu}_p$-torsor.*

(b) Otherwise, the natural transformation (32) is an isomorphism.

Proof: First suppose that $\bar{f}$ is not special, i.e. $\sum_{i\in\mathbb{B}} a_i = p-1$. This implies that $\dim_k H^1(\bar{Z}_0, \mathcal{O})_\chi = 0$. Part (b) now follows from [51, Theorem 4.11].

Suppose that $\bar{f}$ is special, i.e. $\sum_{i\in\mathbb{B}} a_i = \sum_{i\in\mathbb{B}_{\rm ram}} a_i = 2(p-1)$. The differential form ω on $\bar{Z}_0$ corresponds to a line bundle $\bar{\mathcal{L}} \in J(\bar{Z}_0)[p](\bar{k})_\chi$.

The group scheme $J(\bar{Z}_0)[p]$ decomposes into eigenspaces for the H_0-action, since H_0 acts via $\chi(H_0) \subset \mathbb{F}_p^\times$. Write $J(\bar{Z}_0)[p]_\chi$ for the subgroup scheme where H_0 acts via χ. We will show in the proof of Proposition 4.4.1 that we have an exact sequence

$$0 \longrightarrow \boldsymbol{\mu}_p \longrightarrow J(\bar{Z}_0)[p]_\chi \longrightarrow (\mathbb{Z}/p)^{d_{\rm new}+1} \longrightarrow 0,$$

where $d_{\rm new} = |\mathbb{B}_{\rm new}|$. (Section 4.4 is independent of the results of Section 3.) The set of lifts of $\bar{\mathcal{L}}$ to an element of $J(\bar{Z}_{0,R})[p]_\chi$ is a torsor under $\boldsymbol{\mu}_p$. This proves the lemma. □

For every $i \in \mathbb{B}$, we let $\hat{Y}_i$ be the completion of $\bar{Y}_{\rm sing}$ at τ_i. Let $R \in \mathfrak{C}_k$ and $\bar{Y}_R$ be a $\mathcal{G}_0$-equivariant deformation of $\bar{Y}_{\rm sing}$. Write $(\bar{g}_{0,R}, \omega_R)$ for the corresponding deformation datum. Let $i \in \mathbb{B}_{\rm new}$ and choose a point z_i of $\bar{Z}_0$ above $\tau_i \in \bar{X}_0$. Let $H_i \subset H_0$ be the decomposition group of z_i. There exists a local parameter $t = t_i$ of z_i on $\bar{Z}_R$ and a character $\chi_i : H_i \to R^\times$ such that $\mathcal{O}_{\bar{Z}_R, z_i} = R[[t]]$ and $h^* t_i = \chi_i(h) \cdot t_i$ for all $h \in H_i$. We denote by $\hat{Y}_{i,R}$ the completion of $\bar{Y}_R$ at τ_i; this is an equivariant deformation of $\hat{Y}_i$. We obtain a morphism

$$\mathrm{locgl} : \mathrm{Def}\,(\bar{Y}_{\rm sing}, \mathcal{G}_0) \longrightarrow \prod_{i\in\mathbb{B}} \mathrm{Def}\,(\hat{Y}_i, \mathcal{G}_0)$$

called the *local-global morphism* ([51, Section 5.3]). We say that the $\mathcal{G}_0$-equivariant deformation $\bar{Y}_R$ is *locally trivial* if it lies in the kernel of the local global morphism. We denote by

$$\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)^{\mathrm{loctriv}} \subset \mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)$$

the subfunctor parameterizing locally trivial deformations.

Lemma 3.4.2 *The deformation functor* $\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)^{\mathrm{loctriv}}$ *is formally smooth. Its dimension is*

$$\frac{1}{p-1}(\sum_{i\in\mathbb{B}} a_i) - 1.$$

Proof: We use the terminology of [51]. We need to show that $\mathbb{E}\mathrm{xt}^2_{\mathcal{G}_0}(\mathcal{L}_{\bar{Y}_{\mathrm{sing}}/k}, \mathcal{O}_{\bar{Y}_{\mathrm{sing}}}) = 0$. The lemma then follows from [51, Theorem 4.8].

In our situation, the integer $s = \dim_{\mathbb{F}_p} V$ of [51] equals one. This corresponds to the assumption that the order of I_0 is p. This implies that the sheaf $\mathcal{E}xt^1_{\mathcal{G}_0}(\mathcal{L}_{\bar{Y}_{\mathrm{sing}}/k}, \mathcal{O}_{\bar{Y}_{\mathrm{sing}}})$ has support in isolated points (namely the critical points of the deformation datum.) Therefore $H^1(\bar{X}_0, \mathcal{E}xt^1_{\mathcal{G}_0}(\mathcal{L}_{\bar{Y}_{\mathrm{sing}}/k}, \mathcal{O}_{\bar{Y}_{\mathrm{sing}}})) = 0$ and [51, (43)] implies that

$$\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{\bar{Y}_{\mathrm{sing}}/k}, \mathcal{O}_Y) = 0.$$

This implies that the deformation problem is formally smooth.

We now compute the dimension of $\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)^{\mathrm{loctriv}}$. Proposition 4.10 of [51] implies that the tangent space to the deformation functor $\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)^{\mathrm{loctriv}}$ is

$$H^1(\bar{X}_0, \mathcal{H}om(\mathcal{L}_{\bar{Y}_{\mathrm{sing}}/k}, \mathcal{O}_{\bar{Y}_{\mathrm{sing}}})) = H^1(\bar{X}_0, \mathcal{M}^{H_0}),$$

where $\mathcal{M}^{H_0}$ is defined in [51, Section 4.3]. In our situation it is the sheaf of derivations D of $\mathcal{O}_{\bar{X}_0}$ such that $D(\omega)$ is a regular function on $\bar{Z}_0$. The proof of [51, Lemma 5.3] implies that $\mathcal{M}^{H_0}$ is isomorphic to $((\bar{g}_0)_*\mathcal{O}_{\bar{Z}_0})_\chi$. A local calculation shows that

$$\deg(\mathcal{M}^{H_0}) = -\sum_{i\in\mathbb{B}} \frac{a_i}{p-1}.$$

The Riemann–Roch Theorem implies therefore that the dimension of $H^1(\bar{X}_0, \mathcal{M}^{H_0})$ equals $-1 + (\sum_{i\in\mathbb{B}} a_i)/(p-1)$. This proves the lemma. □

In the rest of this section we assume that the deformation datum $(\bar{g}_0, \omega)$ is **special**. Our first goal is to prove the following proposition.

Proposition 3.4.3 *There exists a deformation datum* $(\bar{g}'_0 : \bar{Z}'_0 \to \bar{X}'_0, \omega')$ *with signature* (σ_i) *such that the corresponding pointed curve* $(\bar{X}'_0; \tau_i \,|\, i \in \mathbb{B}_{\mathrm{prim}})$ *is generic.*

Since $r + 1 = 4$, the condition that $(\bar{X}'_0; \tau_i \,|\, i \in \mathbb{B}_{\mathrm{prim}})$ is generic just means that $x_3 = \lambda$ is transcendental over $\mathbb{F}_p$. The proof of Proposition 3.4.3 follows from the deformation theory of $\boldsymbol{\mu}_p$-torsors of [51]. We start by introducing some notation.

Proof: Lemma 3.4.1 implies that $\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)^{\mathrm{loctriv}} \to \mathcal{I}$ is finite and flat, therefore the corresponding deformation spaces have the same dimension, which is one by Lemma 3.4.2. Since the deformation functor $\mathrm{Def}\,(\bar{X}_0; \tau_i \,|\, i \in \mathbb{B}_{\mathrm{prim}})$ also has dimension one, the proposition follows. □

We now come to the accessary parameter problem. Let λ be transcendental over $\mathbb{F}_p$. Proposition 3.4.3 implies that there exists a deformation datum $(\bar{g}'_0, \omega')$ whose signature is the fixed signature (σ_i) and whose set of primitive critical points is $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$ with λ transcendental. Let

$$u' = \prod_{i \in \mathbb{B}_{\rm ns}} (x - \tau_i).$$

By Proposition 3.1.1 there exists accessary parameters $\beta_0, \ldots, \beta_{s-3} \in k((\lambda; \tau_i \mid i \in \mathbb{B}_{\rm new} \cap \mathbb{B}_0))$ such that u' is a solution to the differential equation (27) and residue condition of Proposition 3.1.1.(b) holds. This defines a field extension $k(B_0) := k(\lambda)[\tau_i, \beta_j \mid i \in \mathbb{B}_{\rm new} \cap \mathbb{B}_0,\, 0 \le j \le s-3]$ of $k(\lambda)$. Proposition 3.4.3 implies that this is a finite extension. We let B_0 be the smooth projective curve over k with function field $k(B_0)$, and write $\pi : B_0 \to \mathbb{P}^1_\lambda$ for the natural map. We call this map the *accessary parameter cover*.

Proposition 3.4.4 *The map $\pi : B_0 \to \mathbb{P}^1_\lambda$ is separable.*

If $|\mathbb{B}_0| = |\mathbb{B}_{\rm prim}| = 4$, this follows from the proof of Proposition 3.3.2. In fact, in that case it follows that $\deg(\pi) < p$.

Proof: Define an algebra $A = k(\lambda)[\tau_i, \beta_j \mid i \in \mathbb{B}_{\rm new} \cap \mathbb{B}_0,\, 0 \le j \le s-3]/J$, where $J = (R_i, \rho_j)$, is the ideal expressing the necessary conditions on the τ_i and β_j we encountered in Proposition 3.2.3. These necessary conditions correspond to the fact that the differential equation (27) should have a solution u with $\deg_x(u) = d = (p-1) - (a_0 + \cdots + a_s)/2$ which satisfies the residue condition of Proposition 3.1.1. To prove the proposition, we then need to estimate from above the degree of these relations in the variables. This is similar, though more complicated than what we did in Section 3.3.

We first start by defining the relations $\rho_0, \ldots, \rho_{s-3}$ which express the accessary parameters β_j in terms of the critical points τ_i for $i \in \mathbb{B}_0$. Write

$$P_0 = \sum_{j=1}^{s} \delta_j x^j, \quad P_1 = \sum_{j=0}^{s-1} \epsilon_j x^j, \quad P_2 = \sum_{j=0}^{s-2} \beta_j x^j,$$

with

$$\beta_{s-2} = d(d + a_0), \qquad \delta_s = 1.$$

Write $u = \sum_{i \ge 0} u_i x^i$. It is no restriction to suppose that $u_0 = 1$. As in the proof of Proposition 3.3.2, we obtain a recursion for the coefficients u_i of u from the differential equation $P_2 u'' + P_1 u' + P_0 u = 0$. One computes that for all $i \ge 0$ we have

$$A_i(-1) u_{i+1} + \cdots + A_i(s-2) u_{i-s+2} = 0, \tag{33}$$

where

$$A_i(j) = \beta_j + \epsilon_j (i - j) + \delta_{j+2}(i-j)(i-j-1). \tag{34}$$

In particular,

$$A_i(-1) = (-1)^{s-1}(i+1)(i+1+a_1) \prod_{j=2}^{s} \tau_j, \qquad A_i(s-2) = (i - d - s + 2)(i - d - s - a_0 + 2).$$

We conclude that

$$\deg_{\boldsymbol{\beta}}(A_i(j)) = \begin{cases} 1 & \text{if } 0 \le j \le s-3, \\ 0 & \text{if } j = -1, s-2. \end{cases} \tag{35}$$

Here $\deg_{\boldsymbol{\beta}}$ denotes the total degree in $\beta_0, \ldots, \beta_{s-3}$. This immediately implies the estimate

$$\deg_{\boldsymbol{\beta}}(u_i) \le i. \tag{36}$$

We now describe the relations on the β_j. If $i_0 := p-1-a_1 \le d$ then $A_{i_0}(-1) = 0$. In this case we obtain a first relation

$$A_{i_0}u_{i_0} + \cdots + A_{i_0}(s-2)u_{i_0-s+1} = 0.$$

By (36) and (35) the total degree in β of this relation is less than or equal to i_0+1 which is strictly less than p. As in the proof of Proposition 3.3.2 one should take u_{i_0+1} as a new variable. Whether this case occurs or not does not make any difference in the arguments that follow, therefore we omit the variable u_{i_0+1} from our notation.

First suppose that $d+s-2 < p$. Then the conditions we need to impose on the accessary parameters (β_j) for u to be a solution of the differential equation are

$$u_{d+1} = \cdots = u_{d+s-2} = 0.$$

It follows from the expression for $A_i(s-2)$ that if these conditions are satisfied then $u_{d+s-1} = 0$, and therefore we may take $u_i = 0$ for $i > d$. Therefore we put $\rho_j = u_{d+j+1}$. The assumption on d implies that

$$\deg_{\boldsymbol{\beta}}(u_{d+j}) \le d+s-2 < p, \qquad \text{for } j = 1, \ldots, s-2.$$

Next suppose that $d+s-2 \ge p$. Then $A_{p-1}(-1) = 0$. Since $2d = 2(p-1) - (a_0 + \cdots + a_s)$ and $a_i \ne 0$ it follows that $d < p-1$. Let

$$\rho_j = u_{d+j+1}, \quad \text{for } j = 0, \ldots, p-2-d.$$

As before, we have that

$$\deg_{\boldsymbol{\beta}}(u_{d+j}) \le p-1 < p.$$

For $j = p-1-d$, we have the condition

$$A_{p-1}(0)u_{p-1} + \cdots + A_{p-1}(s-2)u_{p-s+1} = 0.$$

Using that we already imposed $u_{d+1} = \cdots = u_{p-1} = 0$, this conditions may be replaced by

$$\rho_{p-1-d} := A_{p-1}(p-1-d)u_d + \cdots + A_{p-1}(s-2)u_{p-s+1} = 0.$$

Continuing, we define for all $j \ge p-1-d$

$$\rho_j =:= A_{p-1+d-j}(j)u_d + \cdots + A_{p-1+d-j}(s-2)u_{j+d-s+2} = 0$$

whose degree in β is less than or equal to $d+1$ which is strictly less than p.

Next we impose the conditions on the τ_i for $i \in \mathbb{B}_{\rm new}$. Define $Q = \prod_{i\in\mathbb{B}'}(x-\tau_i)^{1+a_i-\nu_i}$. We let R_i be the condition expressing that

$$\mathrm{Res}_{\tau_i}\frac{1}{Qu^2} = 0$$

for $i \in \mathbb{B}_{\rm new} \cap \mathbb{B}_0$. Recall that this condition is automatically satisfied for $i \in \mathbb{B}_{\rm ns}$, see the proof of Proposition 3.1.1. Since the order of $1/Qu^2$ at τ_i is strictly larger than $-p$, it follows that the total degree $\deg_{\boldsymbol{\tau}}(R_i)$ is strictly less than p.

Proposition 3.2.3 implies that we have described all necessary conditions on the (τ_i, β_j) for defining a deformation datum. Therefore the curve B_0 corresponds to a connected component of the normalization of $\mathrm{Spec}(A)$. The degree estimates of the R_i and ρ_j now imply that $\pi : B_0 \to \mathbb{P}^1_\lambda$ is separable. □

An important difference between the version of the accessary parameter problem studied by Dwork and the one of Proposition 3.4.4 is that Dwork considers the question whether the differential equation has nilpotent p-curvature. The existence of the solution u implies that the p-curvature of the differential equation is nilpotent, but in general this appear to be a stronger condition. Namely, we require that the degree of u in x is d which is strictly less than p. It seems possible that the p-curvature is nilpotent, but the differential equation does not admit a solution of degree strictly less than p. It is interesting that this does not occur in case $r+1=s+1=4$. In this case it is shown by Beukers [4] that if the differential equation admits an algebraic solution u, then it admits a polynomial solution of degree strictly less than p. A similar, less precise result in a more general setting is proved by Honda [17]. Honda shows that in general one can give a bound on the degree of a minimal polynomial solutions, if such a solution exists, but this bound is not effective. It seems to me that if $s>3$ such a bound will in general be larger than p. It would be interesting to see if one could rephrase our approach in terms of the p-curvature.

We end this section with a corollary to Proposition 3.4.3. We let $f : Y \to \mathbb{P}^1$ be a G-Galois cover as in the beginning of Section 3.4.

Corollary 3.4.5 *There exists a G-Galois cover $f' : Y' \to \mathbb{P}^1$ with special multiplicative reduction and signature (σ_i) which is branched at $x_0=\infty, x_1=0, x_2=1, x_3=\lambda$ with $(\mathbb{P}^1; x_i)$ generic.*

Proof: This is standard argument using the auxiliary cover construction and formal patching, see for example [50, Section 4.2]. The point is that, by assumption on the cover $f : Y \to \mathbb{P}^1$, there exist G_i-Galois covers $\bar{f}_i : \bar{Y}_i \to \mathbb{P}^1$ with ramification invariant σ_i for every $i \in \mathbb{B}$. Here G_i is a subgroup of G. Recall that $\bar{f}_i$ is wildly branched only at the point $\infty \in \bar{X}_i$ where $\bar{X}_i$ intersects the original component $\bar{X}_0$. The cover $\bar{f}_i$ is unbranched outside ∞ if $i \in \mathbb{B}_{\rm prim}$ and is tamely branched at exactly one other point if $i \in \mathbb{B}_{\rm prim}$. We call such covers *primitive (resp. new) G_i-tail covers* ([50, Definition 2.9]). Since $\sigma_i \leq 2$ for $i \in \mathbb{B}_{\rm new}$ and $\sigma_i \leq 1$ for $i \in \mathbb{B}_{\rm prim}$, all G_i-tail covers with ramification invariant σ_i are locally isomorphic around the unique wild branch point $\infty \in \bar{X}_i$ ([50, Lemma 2.12]).

The proof of the corollary now roughly goes as follows. Let $\bar{f}^{\rm aux} : \bar{Y}^{\rm aux} \to \bar{X}$ be the special fiber of the auxiliary cover of $f : Y \to X$. By Proposition 3.4.3 there exists a locally trivial deformation $\bar{f}'^{\rm aux}$ of $\bar{f}^{\rm aux}$ such that the marked curve $(\bar{X}; \tau_i \,|\, i \in \mathbb{B}_{\rm prim})$ corresponding to $\bar{f}'^{\rm aux}$ is generic. By Proposition 2.4.1 we may "lift" $\bar{f}'^{\rm aux}$ to a G_0-Galois cover $f_R'^{\rm aux} : Y_R'^{\rm aux} \to X_R$ over R (in the sense of Proposition 2.4.1). The local triviality of the G_i-tail cover stated above now implies that we may define a G-Galois cover $f_R' : Y_R' \to X_R$ which agrees with $\mathrm{Ind}_{G_0}^G f_R'^{\rm aux}$ in a neighborhood of the original component $\bar{X}_0$ and such that the restriction of the stable reduction $\bar{f}'$ of f_R' to a tail $\bar{X}_i$ is isomorphic to $\mathrm{Ind}_{G_i}^G \bar{f}_i$. Here one uses formal patching. We refer to [50, Section 4.2] for details. □

Let $\sigma = h/(p-1)$ with $p+1 \leq h \leq 2(p-1)$. It is an interesting question for which σ there exists a group G and a new G-tail cover with ramification invariant σ. The only positive results in this direction that I am aware of are the following. For $\sigma = (p+1)/(p-1)$ there exists a new $\mathrm{PSL}_2(p)$-tail cover. In fact, in this range this is the only possible new $\mathrm{PSL}_2(p)$-tail cover ([13]). For $\sigma = (2p-4)/(p-1)$ there exists a new A_p-tail cover ([9]). For $\sigma = 2$ there exists a new $\mathbb{Z}/p$-tail cover; this is just the Artin–Schreier cover with conductor 2. It would be interesting to know whether all such σ's occur. It seems likely that a closer inspection of the explicit equations written down by Abhyankar yields further results in this direction.

4 The pseudo-elliptic bundle corresponding to a special deformation datum

This section is the heart of the paper. We start by recalling generalities on flat vector bundles and define pseudo-elliptic bundles (Section 4.1). Section 4.2 contains the notation and assumptions which hold for the whole of Section 4. In Section 4.3 we associate to a special deformation datum $(g_k : Z_k \to \mathbb{P}^1_k, \omega)$ an F-crystal V; it is a sub-F-crystal of the de Rham cohomology of a lift of the curve Z_k. We show that $\bar{V} = V \otimes k$ extends to a pseudo-elliptic bundle $(\mathcal{E}, \nabla)$ (Theorem 4.8.2).

A key tool is an explicit description of the differential equation corresponding to the flat vector bundle $(\mathcal{E},\nabla)$ in terms of the Hasse invariant Φ_* and the dual Hasse invariant Φ in Section 4.5. Properties of the Hasse invariant are collected in Section 4.4 which also contains the definition of the supersingular points. (Essentially, these are the zeros of Φ_*). We use the explicit description of $\bar{V}$ to show that the Kodaira–Spencer map is nontrivial (Section 4.7) and that the p-curvature is nilpotent and nonzero (Section 4.8). Theorem 4.8.2 follows from these statements.

A more subtile argument is used in Section 4.7 to show that the Kodaira–Spencer map is an isomorphism, except possibly at the supersingular points which ramify in the accessary parameter cover $\pi : B_0 \to \mathbb{P}^1_\lambda$ (Theorem 4.7.3). This argument uses the deformation theory of $\boldsymbol{\mu}_p$-torsors as in Section 3.4. The condition on the supersingular points is used since an analogous theory for $\boldsymbol{\alpha}_p$-torsors is not available.

The section finishes with some quantitative results on the supersingular points (Section 4.9) and a description of the deformation datum (C_0,θ) corresponding to the pseudo-elliptic bundle $(\mathcal{E},\nabla)$ (Section 4.10). This is an extension of the result of [10]. It introduces the topic of Section 5. Namely, in that section we interprete (C_0,θ) as the Swan conductor of a cover of Hurwitz spaces. Section 4.11 contains a concrete example. For completeness, we give in Section 4.12 some results on the special case of elliptic bundles.

4.1 Pseudo-elliptic bundles In this section we recall generalities on flat vector bundles. We explain the correspondence between flat vector bundles and differential equations. We also define pseudo-elliptic bundles. The following notation replaces the previous notation in this section.

Let k be an algebraically closed field of characteristic $p>0$, and let B_0 be a smooth projective curve over k. We fix $r+1\geq 3$ pairwise distinct points $b_0,\ldots,b_r$ on X, where we suppose that $b_0=\infty$. Denote by $\Omega^{\log}_{B_0/k}=\Omega^1_{B_0/k}(\sum b_i)$ the sheaf of differential 1-forms with at most simple poles in the marked points b_i, and by $\tau^{\log}_{B_0/k}\cong(\Omega^{\log}_{B_0/k})^{-1}$ its dual, i.e. the sheaf of vector fields on B_0 with at least simple zeros in the marked points.

A *flat vector bundle* is a vector bundle $\mathcal{E}$ on B_0 together with a connection $\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega^{\log}_{B_0/k}$. Recall that a *connection* is an additive map

$$\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega^{\log}_{B_0/k}$$

satisfying the Leibniz rule

$$\nabla(fm)=\mathrm{d}f\otimes m+f\nabla(m),$$

for $f\in k(B_0)$ and $m\in\mathcal{E}$. The connection ∇ has regular singularities in the marked points b_i. Since we work on a curve, the connection is automatically integrable.

A *horizontal morphism* from $(\mathcal{E}_1,\nabla_1)$ to $(\mathcal{E}_2,\nabla_2)$ is a morphism $\varphi:\mathcal{E}_1\to\mathcal{E}_2$ of vector bundles which is compatible with the connections. We write $\mathrm{MIC}(B_0)$ for the category of $k(B_0)$-modules with (integrable) connection.

Let $(\mathcal{E},\nabla)$ be a flat vector bundle on B_0. For $i=1,..,r$, we define the *monodromy operator* μ_i as an endomorphism of the fiber $\mathcal{E}|_{b_i}$ of $\mathcal{E}$ at b_i, as follows. Let t be a local parameter at b_i. Then $\nabla(t\partial/\partial t)$ defines a k-linear endomorphism of the stalk $\mathcal{E}_{b_i}$ of $\mathcal{E}$ at b_i which fixes the submodule $\mathfrak{m}_{b_i}\cdot\mathcal{E}_{b_i}$, where $\mathfrak{m}_{b_i}$ denotes the maximal ideal of the local ring $\mathcal{O}_{B_0,b_i}$. Therefore, $\nabla(t\partial/\partial t)$ induces a k-linear endomorphism μ_i of the fiber $\mathcal{E}|_{b_i}=\mathcal{E}_{b_i}/\mathfrak{m}_{b_i}.\mathcal{E}_{b_i}$. One checks easily that μ_i does not depend on the choice of the parameter t.

Let α_i,β_i be the two eigenvalues of μ_i. We call α_i,β_i the *local exponents* of ∇ at b_i. We distinguish two cases. If μ_i is not semisimple, i.e. $\alpha_i=\beta_i$ and

$$\mu_i\sim\begin{pmatrix}\alpha_i & 1\\ 0 & \alpha_i\end{pmatrix},$$

then we say that ∇ has *logarithmic monodromy* at b_i. Otherwise,

$$\mu_i \sim \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix},$$

and we say that ∇ has *toric monodromy* at b_i.

A flat vector bundle $(\mathcal{E}, \nabla)$ corresponds to an ordinary differential equation, as follows. Let $\mathcal{E}^*$ be the K-linear dual of $\mathcal{E}$. We define a connection ∇^* on $\mathcal{E}^*$ by

$$\langle\, \nabla(D)w_1, w_2 \,\rangle + \langle\, w_1, \nabla^*(D)w_2 \,\rangle = D\langle\, w_1, w_2 \,\rangle,$$

for w_1 (resp. w_2) a section of $\mathcal{E}$ (resp. $\mathcal{E}^*$) over an open $U \subset B_0$.

Let e_1 be a section of $\mathcal{E}$ such that $\mathbf{e} = (e_1, e_2 := \nabla(D)e_1)$ forms a basis of $\mathcal{E}$, locally outside the marked points b_i. We call such a basis a *cyclic basis* of $\mathcal{E}$.

Write

$$\nabla(D)\mathbf{e} = A\mathbf{e}, \qquad \text{with} \qquad A = \begin{pmatrix} 0 & -p_2 \\ 1 & -p_1 \end{pmatrix}. \tag{37}$$

Let $\mathbf{e}^*$ be the dual basis of $\mathbf{e}$. Then $\nabla^*(D)\mathbf{e}^* = -A^t\mathbf{e}^*$. Therefore a local section $s = g_1e_1^* + g_2e_2^*$ is horizontal if and only if

$$\begin{cases} g_2 = g_1', \\ L(g_1) := g_1'' + p_1g_1' + p_2g_1 = 0. \end{cases}$$

We call L the *differential operator associated to* $\mathcal{E}$.

Let $b = b_i$ be a marked point and suppose that t is a local parameter at b_i. Then

$$\nabla(t\,\partial/\partial t)(e_1, te_2) = \begin{pmatrix} 0 & -t^2p_2 \\ 1 & 1 - tp_1 \end{pmatrix} (e_1, te_2).$$

Write $p_i = c_it^{-i} + t^{-i+1}(\cdots)$, with $c_i \in k$. The local exponents α_i, β_i are the roots of the so called *indicial equation*:

$$X^2 + (-1 + c_1)X + c_2 = 0. \tag{38}$$

The reason for taking the horizontal vectors of the dual flat vector bundle $\mathcal{E}^*$ is that the concept of local exponents coincides with the classical ones ([17], Appendix).

A *filtration* on a flat vector bundle $(\mathcal{E}, \nabla)$ of rank two consists of a line subbundle $\operatorname{Fil}\mathcal{E} \subset \mathcal{E}$ such that $\operatorname{Gr}\mathcal{E} := \mathcal{E}/\operatorname{Fil}\mathcal{E}$ is also a line bundle. For such a filtration, the connection ∇ induces a *Kodaira–Spencer* map

$$\kappa : \operatorname{Fil}\mathcal{E} \;\longrightarrow\; \operatorname{Gr}\mathcal{E} \otimes \Omega^{\log}_{B_0/k}.$$

If it seems more convenient, we will regard κ as a morphism

$$\kappa : \tau^{\log}_{B_0/k} \;\longrightarrow\; (\operatorname{Fil}\mathcal{E})^{-1} \otimes \operatorname{Gr}\mathcal{E}.$$

Note that, written in either way, κ is $\mathcal{O}_{B_0}$-linear.

We set $\mathcal{T} := (\tau^{\log}_{B_0/k})^{\otimes p}$. This is a line bundle on B_0 of degree $-p(2g - 2 + r) < 0$. We endow $\mathcal{T}$ with the unique connection $\nabla_{\mathcal{T}} : \mathcal{T} \to \mathcal{T} \otimes \Omega^{\log}_{B_0/k}$ such that the subsheaf $\mathcal{T}^{\nabla}$ of horizontal sections consists precisely of the 'p-th powers', i.e. of sections of the form $D^{\otimes p}$, where D is a section of $\tau^{\log}_{B_0/k}$ ([24, Theorem 5.1]).

Let $(\mathcal{E}, \nabla)$ be a flat vector bundle on B_0. The *p-curvature* of $(\mathcal{E}, \nabla)$ is an $\mathcal{O}_{B_0}$-linear morphism

$$\Psi_{\mathcal{E}} : \mathcal{T} \;\longrightarrow\; \operatorname{End}_{\mathcal{O}_{B_0}}(\mathcal{E}),$$

defined as follows. Let D be a rational section of $\tau^{\log}_{B_0/k}$. We regard D as a derivation of the function field $k(B_0)$. Then $D^p := D \circ \cdots \circ D$ is again a derivation of $k(B_0)$ and $\nabla(D)$ and $\nabla(D^p)$ are k-linear endomorphisms of the K-vector space $\bar{V} := \mathcal{E} \otimes_{\mathcal{O}_{B_0}} k(B_0)$. We define

$$\Psi_{\mathcal{E}}(D^{\otimes p}) := \nabla(D)^p - \nabla(D^p).$$

This is a $k(B_0)$-linear endomorphism of $\bar{V}$. One shows that the rule $D^{\otimes p} \mapsto \Psi(D^{\otimes p})$ extends in a unique way to the desired $\mathcal{O}_{B_0}$-linear map $\Psi_{\mathcal{E}}$ ([24, 5.0.1]). One can show that $\bar{V}$ is the reduction mod p of an F-crystal, compare to Section 4.3.

It is important to notice that the p-curvature is *horizontal* in the sense that it commutes with the canonical connections on $\mathcal{T}$ and $\mathrm{End}_{\mathcal{O}_{B_0}}(\mathcal{E})$. Indeed, by the definitions of these connections, the claim that $\Psi_{\mathcal{E}}$ is horizontal is equivalent to the fact that the endomorphisms $\Psi_{\mathcal{E}}(D^{\otimes p})$ and $\nabla(D)$ of $\bar{V}$ commute. This is easy to check, see also [24, 5.2.2].

Definition 4.1.1 A flat vector bundle bundle $(\mathcal{E}, \nabla)$ on B_0 is called

(i) *active*, if $\Psi_{\mathcal{E}} \neq 0$,

(ii) *nilpotent*, if the image of $\Psi_{\mathcal{E}}$ consists of nilpotent endomorphisms,

(iii) *admissible*, if $\Psi_{\mathcal{E}}$ is nonzero at every point $b \in B_0$, except possibly at the marked points.

If $(\mathcal{E}, \nabla)$ is active, a point $b \in B_0$ where $\Psi_{\mathcal{E}}$ vanishes is called a *spike*. We write $n_b := \mathrm{ord}_b(\Psi_{\mathcal{E}})$ for the order of vanishing of $\Psi_{\mathcal{E}}$ at b and say that b is a *spike of order* n_b.

Definition 4.1.2 A *pseudo-elliptic bundle* of B_0 is a flat vector bundle (E, ∇) of rank two which satisfies the following conditions.

(i) There exists a nontrivial filtration $\mathrm{Fil}(\mathcal{E}) \subset \mathcal{E}$ such that the associated Kodaira–Spencer map is nontrivial.

(ii) The flat bundle $(\mathcal{E}, \nabla)$ is active and nilpotent.

A filtration $\mathrm{Fil}\,\mathcal{E} \subset \mathcal{E}$ as in (i) is called a *Hodge filtration*.

The concept of a pseudo-elliptic bundle is a generalization of Ogus' elliptic crystal, see Section 4.4 for a discussion of the differences. It is also a generalization of active, nilpotent indigenous bundles as defined in [10], [35], and [36]. The main difference is that the Kodaira–Spencer map of an indigenous bundle is required to be an isomorphism, rather than just nonzero. We decided to introduce this new notion here since we were not able to show that the flat vector bundles we define are always isogenous. Moreover, for the application to the reduction of Hurwitz spaces this is not an essential property.

The p-curvature $\Psi = \Psi_{\mathcal{E}}$ is p-linear, [24, Proposition 5.2]. To check whether Ψ is nilpotent it suffices to check the condition for one derivation D. The notion of p-curvature coincides with the classical notion as in [15]. It is shown in [15, Section 2.1] that Ψ is nilpotent if and only if $\Psi(D)$ is a nilpotent matrix, for some derivation D. Moreover, Ψ is active if and only if $\Psi(D)$ is nonzero as element of $M_2(k(B_0))$. Note that Ψ is nilpotent if and only if the p-curvature Ψ^* of the dual module (M^*, ∇^*) is nilpotent.

It is shown by Honda ([17, Appendix]) that Ψ is nilpotent if and only if L has *sufficiently many solutions in a weak sense*. This means that the differential equations

- $L(y) = y'' + p_1 y' + p_2 y = 0,$
- $L_W(w) := w' + p_1 w = 0.$

both have an algebraic solution. The first order differential equation $L_W(w) = 0$ is called the *Wronskian equation*. Let g_1, g_2 be solutions of $L(g) = 0$. Then the Wronskian $W := W(g_1, g_2) := g_1 g_2' - g_1' g_2$ satisfies $L_W(W) = 0$.

Let us, from now on, assume that $(\mathcal{E}, \nabla)$ is active and nilpotent, and choose some rational section D of $\tau^{\log}_{B_0/k}$. Let $\mathcal{M} \subset \mathcal{E}$ be the kernel of $\Psi_{\mathcal{E}}(D^{\otimes p})$, i.e. the maximal subbundle on which $\Psi_{\mathcal{E}}(D^{\otimes p})$ is zero. Our assumption implies that $\mathcal{M}$ is a saturated line bundle. Let $\mathcal{L} := \mathcal{E}/\mathcal{M}$

denote the quotient line bundle. It follows from the fact that $\Psi_{\mathcal{E}}$ is horizontal that $\mathcal{M}$ is invariant under the connection ∇. In other words, we obtain a short exact sequence of flat vector bundles

$$0 \to \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \to 0.$$

Moreover, the p-curvature of the induced connections on $\mathcal{M}$ and $\mathcal{L}$ is zero. If $(\mathcal{E}, \nabla)$ is a pseudo-elliptic bundle, the Kodaira–Spencer map is nonzero. This implies that $\mathcal{M}$ is a complement to $\mathrm{Fil}(\mathcal{E})$.

4.2 The setup Suppose we are given a special deformation datum (g, ω) with $|\mathbb{B}_0| = s+1 \geq |\mathbb{B}_{\mathrm{prim}}| = r+1 = 4$. Let (σ_i) be the signature of the deformation datum and (τ_i) the set of critical points. Recall that we write $\sigma_i = \nu_i + a_i/(p-1)$ with $0 \leq a_i < p-1$ and $\nu_i \geq 0$. Assume that $\mathbb{B}_{\mathrm{wild}} = \{i \in \mathbb{B} \mid a_i = 0\} = \emptyset$. Together with the assumption that the deformation datum is special this implies that $a_i \neq 0$ for $i = 0, \ldots, s$. We suppose that $\mathbb{B}_{\mathrm{prim}} = \{0,1,2,3\}$ and that $\tau_1 = 0, \tau_2 = 1, \tau_3 = \lambda, \tau_0 = \infty$, where $(\bar{X}_0; \tau_i)$ is generic. We choose a subset $\mathbb{B}_{\mathrm{ns}} \subset \{i \in \mathbb{B}_{\mathrm{new}} \mid \sigma_i = (p+1)/(p-1)\}$ of nonsingular critical points.

Recall that to the deformation datum (g, ω) we associated a curve Z_k defined as a connected component of the smooth projective curve given by the equation

$$z^{p-1} = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3}u^2. \tag{39}$$

The cover $g_k : Z_k \to \mathbb{P}^1_k$ sends (x, z) to x. The differential ω is a certain multiple of

$$\omega_0 = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)}. \tag{40}$$

Proposition 2.3.3 implies that the deformation datum is multiplicative. This means that ω is generically logarithmic. The polynomial $u \in k(B_0)[x]$ is the solution of a certain Fuchsian differential equation (27). Here $\pi : B_0 \to \mathbb{P}^1_\lambda$ is the cover defined by the accessary parameter (Section 3.4). The coefficients of $u = u(x)$ depend on λ and the accessary parameters $\beta_0, \ldots, \beta_{s-3}$. We showed in Proposition 3.4.4 that $\pi : B_0 \to \mathbb{P}^1_\lambda$ is finite and separable. In this section, we will mostly consider λ as "parameter" on B_0. This makes sense outside the ramification locus of $B_0 \to \mathbb{P}^1_\lambda$, and is mostly very convenient. However, for doing local computations at the ramification locus one may have to change the parameter.

Recall from Section 2.3 that to a special deformation datum we may associate the Hasse invariant Φ_* and the dual Hasse invariant Φ. These are elements of $k(B_0)$.

In the rest of this paper we make the following assumption.

Assumption 4.2.1 (a) The deformation datum (g, ω) is special (Definition 2.3.1) and $\mathbb{B}_{\mathrm{wild}} = \emptyset$.

(b) The dual Hasse invariant Φ is nonzero as function of $k(B_0)$.

As discussed in Section 2.3, we may reformulate Assumption 4.2.1 in terms of the Hodge filtration on Z_k, as follows. The dimension of $H^1(Z_k, \mathcal{O})_\chi$ as k-vector space is 1, and $F : H^1(Z_k, \mathcal{O})_\chi \to H^1(Z_k, \mathcal{O})_\chi$ is generically an isomorphism.

The following proposition determines for which values of λ the curve Z_k is singular.

Proposition 4.2.2 *(a) Write $u(x) = \prod_{i \in \mathbb{B}_{\mathrm{ns}}}(x - \tau_i)$. For every $b \in B_0 - \pi^{-1}(\{0,1,\infty\})$ the zeros of $u(b)(x)$, different from $x = 0, 1$, are simple.*

(b) If $b \in B_0 - \pi^{-1}(\{0,1,\infty\})$ then $\tau_i(b)$ and $\tau_j(b)$ for $i, j \in \mathbb{B}_{\mathrm{ns}}$ are different if $i \neq j$ and $\tau_i \neq \{0, 1, \infty, \lambda(b)\}$.

Proof: Part (a) follows immediately from the fact that u is the solution to a Fuchsian differential equation (23). Part (b) is more or less the same thing. Suppose that for b with $\pi(b) \neq 0, 1, \infty$ there exists distinct i, j such that $\tau_i(b) = \tau_j(b)$. Then $u(b)(x)$ has a double zero at $\tau_i = \tau_j$. But by (a), this is only possible if $\tau_i = \tau_j \in \{0, 1, \infty, \lambda(b)\}$. □

Proposition 4.2.2 implies that the curve Z_k is singular for $\lambda = 0, 1, \infty$ and for the points $b \in B_0$ for which $\tau_i(b) \in \{0, 1, \infty, \lambda\}$, for some $i \in \mathbb{B}_{\rm new}$. We denote this set by $\Sigma_0 \subset B_0$. We will show that Σ_0 is the set of singularities of $(\bar{V}, \nabla)$. We choose once and for all a lift B of B_0 over $W(k)$ and let Z/B be a lift of Z_k over B. (For example, we may define Z by the "same" equation (39).)

4.3 The F-crystal associated to a deformation datum Notations and assumptions are as in Section 4.2. The first step in associating to the special deformation datum (g, ω) a pseudo-elliptic bundle $(\mathcal{E}, \nabla)$ is to define the F-crystal $\bar{V} := \Gamma(B_0, \mathcal{E})$. This is the goal of this section.

We use the notation of Section 2.4. In particular, we suppose that we have chosen a lift $x_i \in \mathbb{P}^1(K_0)$ of τ_i, where K_0 is an unramified extension of $\mathbb{Q}_p$. We defined a G_0-Galois cover $f_{K_0} : Y_{K_0} \to X_{K_0}$ branched at the x_i which has stable reduction over a finite extension K/K_0. The stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ defines the special deformation datum $(\bar{g}_0, \omega)$. The cover f_{K_0} factors through the H_0-Galois cover $Z_{K_0} \to X_{K_0}$ which is branched at the x_i.

By definition, the field K_0 is an extension of $\mathbb{Q}_p$ of transcendence degree one. Concretely, K_0 is a finite extension of $\mathbb{Q}_p(\lambda)$. Therefore K_0 is the function field of a smooth projective curve B which admits a finite cover $\pi : B \to \mathbb{P}^1_\lambda$. Write $\Sigma_0 \subset \mathbb{P}^1_\lambda$ for the locus where Z_{K_0} is singular. This locus contains the points of B above $\{0, 1, \infty\} \subset \mathbb{P}^1_\lambda$, but is, in general, larger. For every $t \in \mathbb{P}^1_\lambda(K_0) - \{0, 1, \infty\}$, we obtain a specialized cover $Z_t \to X_t$ which is branched at $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = t, x_4(t), \ldots, x_{3+d_{\rm new}}(t)$. By construction, the curve B has good reduction to characteristic p; we denote its special fiber by B_0.

Choose $b \in B$ such that b does not reduce to a point of $\Sigma_0 \subset B_0$. Write $g_b : Z_b \to X_b$ for the fiber of $g : Z \to X$ at b, and let $\bar{g}_b : \bar{Z}_b \to \bar{X}_b$ be it stable reduction. Then there is a unique irreducible component $\bar{Z}_{0,b}$ of $\bar{Z}_b$ above the original component $\bar{X}_{0,b}$ of $\bar{X}$. The genus of $\bar{Z}_0$ equals the genus of Z, and we may identify $g_k : Z_k \to \mathbb{P}^1_k$ with the restriction $\bar{g}_{0,b} : \bar{Z}_{0,b} \to \bar{X}_{0,b}$.

Recall that we defined rational functions $\Phi_*, \Phi \in k(B_0)$ (Section 2.3). The assumption that our deformation datum is multiplicative implies that Φ_* is nonzero. We require that Φ is nonzero also (Assumption 4.2.1). We choose arbitrary lifts of Φ_* and Φ to K_0 which we denote again by Φ_* and Φ.

Define $R_{\rm ord}$ as the p-adic completion of $W(k(B_0))(1/\Phi\Phi_* \prod_{b \in \Sigma_0}(\lambda - b))$. Put $\mathcal{S}_{\rm ord} = {\rm Spec}(R_{\rm ord})$. We obtain an inclusion of F-crystals

$$H^1_{\rm cris}(\bar{Z}_0/\mathcal{S}_{\rm ord}) \subset H^1_{\rm cris}(\bar{Z}/\mathcal{S}_{\rm ord}) \simeq H^1_{\rm dR}(Z_R/\mathcal{S}_{\rm ord}).$$

We write $M_\chi = H^1_{\rm cris}(\bar{Z}_0/\mathcal{S}_{\rm ord})_\chi$ and consider it as sub-F-crystal of $H^1_{\rm dR}(Z_R/\mathcal{S}_{\rm ord})_\chi$. Write $\bar{M}_\chi = M_\chi \otimes_{R_{\rm ord}} \mathbb{F}_p$. Write ${\rm Fil}^1$ for the ${\rm Fil}^1$-part of the Hodge filtration of M_χ.

Lemma 4.3.1 *There exists a sub-F-crystal U called the unit root sub-F-crystal of M_χ such that*

$$M_\chi = {\rm Fil}^1 \oplus U.$$

Proof: This follows from [25], completely analogous to the results in Section 1.3. □

Choose a basis vector η of $U/\mathcal{S}_{\rm ord}$. Write $F\eta = G\eta$ and $\nabla\eta = -H\eta \otimes {\rm d}\lambda$. As in Lemma 1.3.5, we may assume that $G \equiv \Phi \bmod p$. Since U is an F-crystal, the following diagram commutes

$$\begin{array}{ccc} \varphi^* U & \xrightarrow{\quad F \quad} & U \\ \Big\downarrow{\scriptstyle \varphi^*\nabla} & & \Big\downarrow{\scriptstyle \nabla} \\ \varphi^* U \otimes \Omega^1_{\mathcal{S}_{\rm ord}} & \xrightarrow{F \otimes {\rm Id}} & U \otimes \Omega^1_{\mathcal{S}_{\rm ord}}. \end{array} \qquad (41)$$

This implies that

$$G' - HG = -p\lambda^{p-1} H^{\varphi} G.$$

In particular $H \equiv G'/G \equiv \Phi'/\Phi \bmod p$.

Proposition 4.3.2 *(a) There exists a holomorphic differential $\omega_0 \in M_\chi$ such that*

$$F\varphi^*\omega_0 = pD_0\omega_0 + pD_1\eta,$$

for some $D_0, D_1 \in R_{\mathrm{ord}}$.

(b) The elements ω_0 and η span a sub-F-crystal V_χ of M_χ of rank two.

Proof: Recall that $\mathcal{C}\bar{\omega}_0 = \Phi_*^{1/p}\bar{\omega}_0$ and $\Phi_* \neq 0$, by assumption. Therefore there exists $\bar{\omega}_1, \ldots, \bar{\omega}_d \in H^0(\bar{Z}_0, \Omega^1)_\chi = \overline{\mathrm{Fil}}^1 \subset \bar{M}_\chi$ which span a d-dimensional complement to $\bar{\omega}_0$ which is stable under $\mathcal{C}$. Write $\bar{\eta}$ for the image of η in $\bar{M}_\chi$.

In [33] it is shown that $F(\mathrm{Fil}^1) \subset pM_\chi$. Since the restriction of F to Fil^1 is divisible by p, we may define $\phi^1 = F/p : \mathrm{Fil}^1 \to \bar{M}_\chi$. It follows from [20, Proposition 3.8.c] that the composition of ϕ^1 with the projection of $\bar{M}_\chi$ to $\overline{\mathrm{Fil}}^1$ is given by the inverse of the Cartier operator. Therefore, the matrix (modulo p) of ϕ^1 with respect to our basis is

$$\phi^1 \equiv \begin{pmatrix} \Phi_*^{-1} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \psi & \\ 0 & & & \\ e_0 & e_1 & \ldots & e_d \end{pmatrix}, \tag{42}$$

for certain coefficients e_i.

By approximating modulo higher and higher power of p and using that ψ is invertible, one checks that there exists a lift ω_0 of $\bar{\omega}_0$ such that $F\omega_0 = pD_0\omega_0 + pD_1\eta$, for some $D_0, D_1 \in R$. Moreover, we may choose lifts $\omega_1, \ldots, \omega_d$ of $\bar{\omega}_1, \ldots, \bar{\omega}_d$ such that F stabilizes the subspace spanned by $\omega_1, \ldots, \omega_d$. In particular, this shows that $F\omega_0 = pD_0\omega_0 + pD_1\eta$, for some functions $D_0, D_1 \in R_{\mathrm{ord}}$. (The argument we use here is essentially the same as in the definition of the unit root crystal.)

Let V_χ be the subspace of M_χ spanned by ω_0 and η. To show that V_χ is a sub-F-crystal, we have to show that V_χ is stable under the Frobenius morphism and the connection. We already know that F stabilizes V_χ. We claim that this automatically implies that ∇ stabilizes V_χ also. Write

$$\nabla(\frac{\partial}{\partial\lambda})\,\omega_0 = C_0\omega_0 + \sum_{i=1}^{d} C_i\omega_i + C_{d+1}\eta,$$

with $C_i \in R$. Since $M_\chi = \mathrm{Fil}^1 \oplus U$ is an F-crystal, we have a commutative diagram

$$\begin{array}{ccc} \varphi^*\mathrm{Fil}^1 & \xrightarrow{\phi^1} & M_\chi \\ \downarrow{\scriptstyle \varphi^*\nabla} & & \downarrow{\scriptstyle \nabla} \\ \varphi^*\mathrm{Fil}^1 \otimes \Omega^1_{\mathcal{S}_{\mathrm{ord}}} & \xrightarrow{\frac{1}{p}F\otimes\mathrm{Id}} & M_\chi \otimes \Omega^1_{\mathcal{S}_{\mathrm{ord}}}. \end{array} \tag{43}$$

We apply this to ω_0 and compute that

$$\nabla \circ \phi^1\varphi^*(\omega_0) = [(D_0' + D_0C_0)\omega_0 + D_0\sum_{i=1}^{d} C_i\omega_i + (D_0C_{d+1} + D_1' - D_1H)\eta] \otimes \mathrm{d}\lambda.$$

Note

$$\varphi^*\nabla(\omega_0) = (C_0^\varphi\omega_0 + \sum_{i=1}^{d} C_i^\varphi\omega_i + C_{d+1}^\varphi\eta)\otimes \mathrm{d}\lambda^p.$$

This implies that

$$\frac{1}{p}(F\otimes \mathrm{Id})\circ\varphi^*\nabla(\omega_0) \equiv \lambda^{p-1}C_{d+1}^\varphi G\eta\otimes \mathrm{d}\lambda \pmod{p}.$$

The commutativity of (43) implies therefore that $p|C_i$ for $i = 1,\dots,d$. Writing $C_i = p\delta_i^1$ and repeating the argument shows that $p^2|C_i$ for all $i = 1,\dots,d$. Continuing, we find that $C_i = 0$ for all $i = 1,\dots,d$. This shows that ∇ stabilizes V_χ and therefore that V_χ is an F-crystal. □

For future reference, we note that since $C_0 \equiv -D_0'/D_0 \bmod p$ and $D_0 \equiv 1/\Phi_*$ we conclude that

$$C_0 \equiv \frac{\Phi_*'}{\Phi_*} \pmod{p}.$$

4.4 The Hasse invariant In Section 4.9 we define the Hasse invariant and the dual Hasse invariant in the context of filtered flat vector bundles. Essentially, they correspond to the functions Φ_* and Φ on B_0 defined in Section 2.3. By abuse of notation, we will also call Φ_* (resp. Φ) the Hasse invariant (resp. the dual Hasse invariant.) The goal of this section is to prove some properties of Φ_* and Φ which play an important role in the description of the differential equation corresponding to $(\mathcal{E},\nabla)$ in Section 4.5.

Recall that Φ_* and Φ are defined as certain expansion coefficients of

$$\omega_0 = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)} \in H^0(\bar{Z}_0,\Omega)_\chi, \qquad \text{and} \qquad \omega_{0,*} = \frac{\mathrm{d}x}{z} \in H^0(\bar{Z}_0,\Omega)_{\chi^{-1}}.$$

Write $u = \sum_{i=0}^d u_i x^i$. Then for $i = 0,\dots,d$, we have that $\Phi_* u_i^p$ is the coefficient of $x^{p(i+1)-1}$ in $x^{p-1-a_1}(x-1)^{p-1-a_2}(x-\lambda)^{p-1-a_3}u^{p-2}$ and Φ is the coefficient of x^{p-1} of $x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3}u^2$. Recall that we assume that Φ is nonzero as elements of $k(B_0)$ (Assumption 4.2.1). We also know that Φ_* is nonzero as element of $k(B_0)$ (Proposition 2.3.3).

We can characterize Φ and Φ_* in terms of the action of the Cartier operator on differential forms. Namely, we have that

$$\mathcal{C}\omega_0 = \Phi_*^{1/p}\omega_0, \qquad \mathcal{C}\omega_{0,*} = \Phi^{1/p}\omega_{0,*}.$$

It follows that $\omega = \Phi_*^{1/(p-1)}\omega_0$ (resp. $\omega_* := \Phi^{/(p-1)}\omega_{0,*}$) are fixed by the Cartier operator, hence are logarithmic differentials. (Here we use Assumption 4.2.1.(c).) Alternatively, one can also describe Φ, Φ_* in terms of the action of the Frobenius morphism on $H^1(\bar{Z}_0,\mathcal{O})$.

We denote by $U_{\mathrm{ord}} \subset B_0 - \Sigma_0$ the locus where $\mathrm{ord}_b(\Phi_*) \equiv 0 \bmod p$ and $\mathrm{ord}_b(\Phi) \equiv 0 \bmod p$. We call U_{ord} the *ordinary locus*. To $\bar{V} = V\otimes\mathbb{F}_p$, we may associate a flat vector bundle $\mathcal{E}$ on $U_{\mathrm{ord}} \subset B_0 - \Sigma_0$. This vector bundle together with the filtration associated by the Hodge filtration on the sheaf $\mathcal{H}^1_{\mathrm{dR}}(Z/B)$ defines a lat vector bundle $\mathcal{E}$ on U_{ord}. We will show that $\mathcal{E}$ extends to a flat vector bundle on B (Proposition 4.6.1).

It follows from the definition of V, that we have an exact sequence

$$0 \to \mathrm{Fil} \to \bar{V} \to H^1(\bar{Z}_0,\mathcal{O})_\chi \to 0,$$

where Fil is the 1-dimensional subspace of $H^0(\bar{Z}_0,\Omega)_\chi$ spanned by ω_0. Over the ordinary locus, we have a splitting $\bar{V} = \mathrm{Fil}\oplus\bar{U}$, where $\bar{U}$ is the reduction mod p of the unit root part (Section 1.3). Since the unit root part U is an F-crystal, the Frobenius morphism induces an isomorphism $F : \bar{U} \to \bar{U}$. In Section 4.3 we constructed a generator η of $\bar{U}$ which satisfies $F\eta = \Phi\eta$. On Fil, the Frobenius morphism is divisible by p. Therefore we have a map $\phi^1 = F/p : \mathrm{Fil} \to \bar{V}$. We computed that $\phi^1\omega = \Phi_*^{-1}\omega_0 + e_0\eta$, for some function e_0 (42).

Proposition 4.4.1 *Let $b \in B_0 - \Sigma_0$. If $\mathrm{ord}_b(\Phi_*) \not\equiv 0 \bmod p$, then $\mathrm{ord}_b(\Phi) \not\equiv 0 \bmod p$ as well.*

Proof: This proposition is well known if $u = 1$, and in that case the converse holds also. (This is the situation of Example 2.1.2. It is proved for example in [8, Proposition 2.7].) We write H_0 for the Galois group of $\bar{g}_0$. Recall that H_0 has order prime to p. The argument we give here is adapted from the proof of [49, Lemma 1.4].

We write $J := J(\bar{Z}_0)$. For every $b \in B_0 - \Sigma_0$, we write J_b for the fiber of J at b. Similarly, we write J_{gen} for the generic fiber. The group scheme $J[p]$ decomposes into eigenspaces for the H_0-action, since $\mathbb{F}_p$ contains the $(p-1)$th roots of unity and H_0 acts on $J[p]$ via its image $\chi(H_0) \subset \mathbb{F}_p^\times$. Write

$$J[p] = \prod_i J[p]_{\chi^i}$$

for this eigenspace decomposition.

By the comparison isomorphism between de Rham cohomology and crystalline cohomology, it follows that $H^1_{\mathrm{dR}}(\bar{Z}_0)_\chi$ defines the (contravariant) Dieudonné module of the finite flat group scheme $J[p]_\chi$. It follows that the sub-F-crystal $\bar{V} \subset H^1_{\mathrm{dR}}(\bar{Z}_0)_\chi$ correspond to a finite flat group scheme $\mathcal{G}$ of rank p^2 which is a quotient of $J[p]$.

We claim that for $b \in U_{\mathrm{ord}}$ we have an isomorphism $\mathcal{G} \simeq \mathbb{Z}/p \times \boldsymbol{\mu}_p$. It is well known that a holomorphic logarithmic differential corresponds to a p-torsion point. The natural isomorphism

$$H^0(\bar{Z}_0, \Omega)^{\mathcal{C}} \to J[p]$$

is compatible with the H_0-action. Therefore it induces an isomorphism on the χ-eigenspaces. By Serre duality, we obtain an isomorphism

$$H^1(\bar{Z}_0, \mathcal{O})^F_\chi \to \mathrm{Hom}(\boldsymbol{\mu}_p, J[p])_\chi,$$

where Hom should be regarded in the category of finite flat group schemes.

Recall that $\omega \in \bar{V}$ is a holomorphic and logarithmic differential, i.e. $\omega \in H^0(\bar{Z}_{0,b}, \Omega)^{\mathcal{C}}_\chi$ for $b \in U_{\mathrm{ord}}$. Therefore ω corresponds to a p-torsion point $P \in J[p]_\chi$ over U_{ord}. Similarly, via Serre Duality, ω_* corresponds to a map $\boldsymbol{\mu}_p \hookrightarrow J[p]_\chi|_{U_{\mathrm{ord}}}$. The group scheme $\mathcal{G} \subset J[p]_\chi|_{U_{\mathrm{ord}}}$ is generated by the image of ω and ω_*. In particular, $\mathcal{G} \simeq \mathbb{Z}/p \times \boldsymbol{\mu}_p$. We write $\mathcal{G}^{\mathrm{D}} \subset J[p]_{\chi^{-1}}|_{U_{\mathrm{ord}}}$ for the dual group scheme.

Since $\dim H^1(\bar{Z}_0, \mathcal{O})_\chi = 1$ and $\dim H^0(\bar{Z}_0, \Omega)_\chi = s + d - 2$, it follows that $J[p]_\chi$ has rank p^{s+d-1}. Choose $b \in U_{\mathrm{ord}}$. Assumption 4.2.1 implies that this holds for b in a dense open subset of B_0. The definition of Φ implies that $F : H^1(\bar{Z}_0, \mathcal{O})_\chi \to H^1(\bar{Z}_0, \mathcal{O})_\chi$ is an isomorphism. The above identifications imply that the étale part of $J_b[p]_{\chi^{-1}}$ has rank p. Therefore, after passing to the separable closure we may write

$$J_b[p]_{\chi^{-1}} \simeq \mathcal{G}^{\mathrm{D}} \times (\boldsymbol{\mu}_p)^{n(b)} \times \Lambda(b),$$

where $\Lambda(b)$ is a local-local group scheme. Dualizing, we find that

$$J_b[p]_\chi \simeq \mathcal{G} \times (\mathbb{Z}/p)^{n(b)} \times \Lambda(b)^{\mathrm{D}}. \tag{44}$$

There exists a canonical isomorphism

$$\mathrm{Lie}(J[p]) = \mathrm{Lie}(J) \simeq H^1(\bar{Z}_0, \mathcal{O}),$$

([37, p. 147]), which is compatible with the H_0-action. This implies that $\mathrm{Lie}(J[p])_\chi \simeq H^1(\bar{Z}_0, \mathcal{O})_\chi$ is 1-dimensional. Therefore (44) implies that $\Lambda(b)^{\mathrm{D}}$ is trivial. This means that

$$J_b[p]_\chi \simeq \mathcal{G} \times (\mathbb{Z}/p)^{s+d-3} \simeq \boldsymbol{\mu}_p \times (\mathbb{Z}/p)^{s+d-2}.$$

Now let $b \in B_0 - \Sigma_0$ be such that $\mathrm{ord}_b(\Phi_*) \not\equiv 0 \bmod p$. We want to show that $\mathrm{ord}_b(\Phi) \not\equiv 0 \bmod p$ as well. As before, we may write

$$J_b[p]_\chi \simeq (\mathbb{Z}/p)^{n(b)} \times (\boldsymbol{\mu}_p)^{m(b)} \times \Lambda(b)^{\mathrm{D}},$$

where $\Lambda(b)$ is a local-local group scheme.

We know that $H^1(\bar{Z}_0, \mathcal{O})_\chi$ is 1-dimensional, and that $F : H^1(\bar{Z}_{0,b}, \mathcal{O})_\chi \to H^1(\bar{Z}_{0,b}, O)_\chi$ is an isomorphism if and only if $\mathrm{ord}_b(\Phi) = 0 \bmod p$. Therefore $m(b) = 0$ if $\mathrm{ord}_b(\Phi) \not\equiv 0 \bmod p$ and $m(b) = 1$ otherwise. Moreover, the assumption $\mathrm{ord}_b(\Phi_*) \not\equiv 0 \bmod p$ implies that $\Lambda(b)^{\mathrm{D}}$ is nonzero. In particular, $\dim \mathrm{Lie}(\Lambda(b)^{\mathrm{D}})$ is nonzero. Therefore $\dim \mathrm{Lie}(J_b[p])_\chi \leq 1$ implies that $m(b) = 0$, and hence that $\mathrm{ord}_b(\Phi) \not\equiv 0 \bmod p$. □

Corollary 4.4.2 *Let $b \in B_0 - U_{\mathrm{ord}}$. Then one of the following occurs:*

- $\mathcal{G}_b \simeq \mathbb{Z}/p \times \boldsymbol{\alpha}_p$,
- $\mathcal{G}_b$ *is a local-local group scheme.*

Proof: Suppose that $b \in B_0 - U_{\mathrm{ord}}$. Proposition 4.4.1 implies that $\mathrm{ord}_b(\Phi) \not\equiv 0 \bmod p$. The statement of the proposition now follows immediately from the relation between $\mathcal{G}_b$ and the Hasse invariants Φ and Φ_* as explained in the proof of Proposition 4.4.1. The two cases correspond to $\mathrm{ord}_b(\Phi_*) \not\equiv 0 \bmod p$ and $\mathrm{ord}_b(\Phi_*) \equiv 0 \bmod p$. □

Definition 4.4.3 Define $\Sigma_1 \subset B_0 - \Sigma_0$ be be the set of points b such that $\mathrm{ord}_b(\Phi_*) \not\equiv 0 \bmod p$. We call these points the *supersingular points* of the deformation datum.

Corollary 4.4.2 states that $b \in B_0 - \Sigma_0$ is supersingular if and only if $\mathcal{G}_b$ is a local-local group scheme.

The converse to Proposition 4.4.1 does not hold. Corollary 4.7.4 describes the points $b \in B_0$ for which the group scheme $\mathcal{G}_b$ is isomorphic to $\mathbb{Z}/p \times \boldsymbol{\alpha}_p$. A concrete example is given in Section 4.11. Ogus defines in [38] elliptic crystals. These are certain 2-dimensional F-crystals very similar to our crystal $\bar{V}$. As in our case the group scheme $\mathcal{G}$ corresponding to an elliptic crystal is generically isomorphic to $\mathbb{Z}/p \times \boldsymbol{\mu}_p$. However Ogus does not allow $\mathcal{G}_b \simeq \mathbb{Z}/p \times \boldsymbol{\alpha}_p$.

4.5 Explicit description of the crystal $\bar{V}$ In this section we compute the differential equation corresponding to the F-crystal $\bar{V}$ in terms of the Hasse invariant Φ_* and the dual Hasse invariant Φ. In some sense, this is a concrete version in our situation of the result of Katz [28]. In fact, the result of Katz implies that one can extend this description to the whole F-crystal V, by using the higher expansion coefficients. These higher expansion coefficients are analogs of the polynomials $B_n(i)$ of Section 1.3.

Lemma 4.5.1 *Write $\omega_0' := \nabla(\partial/\partial\lambda)\omega_0$. The image of $\lambda(\lambda-1)\omega_0'$ in $H^1(\bar{Z}_0, \mathcal{O})_\chi$ equals $\xi = z/x$.*

Proof: We deduce this lemma from Lemma 1.1.3.

Write

$$u = \prod_{i \in \mathbb{B}_{\mathrm{ns}}} (x - \tau_i), \qquad u_0 = \prod_{\mathbb{B}_0 \cap \mathbb{B}_{\mathrm{new}}} (x - \tau_i).$$

One computes that

$$\omega_0' = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)u_0 u} \left(\frac{(1+a_3)u_0 u}{x - \lambda} + u \sum_{i \in \mathbb{B}_0 \cap \mathbb{B}_{\mathrm{new}}} \frac{a_i}{x - \tau_i} \frac{\partial \tau_i}{\partial \lambda} - 2u_0 \frac{\partial u}{\partial \lambda} \right). \qquad (45)$$

Note that $\partial u/\partial\lambda$ and $\partial u_0/\partial\lambda$ have poles in the ramification points of $\pi : B_0 \to \mathbb{P}^1_\lambda$. However, this makes no difference for our argument.

To be able to apply Lemma 1.1.3, we need to find a representative $\tilde{\omega}_0$ of the class of ω_0' in $H^1_{\mathrm{dR}}(Z/B)_\chi \otimes \mathbb{F}_p$ which has no pole outside $x = \infty$. One computes that

$$\mathrm{d}\left(\frac{z}{x-\lambda}\right) = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)u_0 u}\left(-\frac{(1+a_3)\lambda(\lambda-1)u_0(x=\lambda)u(x=\lambda)}{x-\lambda} + \text{holomorphic}\right).$$

Therefore

$$\lambda(\lambda-1)\tilde{\omega}_0 := \lambda(\lambda-1)\omega_0' + \mathrm{d}\frac{z}{x-\lambda}$$

is holomorphic outside $x = \infty$. Moreover, $z/(x-\lambda)$ is regular outside $x = \lambda, \infty$. Therefore $[\tilde{\omega}_0] = [\omega_0'] \in H^1_{\mathrm{dR}}(Z/B)_\chi$.

We claim that $\lambda(\lambda-1)\tilde{\omega}_0 - \mathrm{d}z/x$ is holomorphic outside $x = 0$. The lemma follows from this claim and Lemma 1.1.3.

To prove the claim, we note that the image of $\lambda(\lambda-1)\omega_0'$ in $H^1(\bar{Z}_0, \mathcal{O})_\chi$ is nonzero, since ω_0' is a differential of the second kind which is not holomorphic. This implies that there exists a constant e such that $\lambda(\lambda-1)\tilde{\omega}_0 - e\,\mathrm{d}(z/x)$ is holomorphic outside $x = 0$. Write

$$\lambda(\lambda-1)\tilde{\omega}_0 = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)u}\sum_{i=0}^{d+s-1} g_i x^i \quad \text{and} \quad \mathrm{d}\frac{z}{x} = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)u}\sum_{i=-\infty}^{d+s-1} h_i x^i.$$

One computes that $g_{d+s-1} = h_{d+s-1} = (-1-a_1-a_2-a_3-2d)$. This proves the claim, and hence the lemma. □

Proposition 4.5.2 *Let ρ be a parameter on B_0 and $D = \partial/\partial\rho$. Write*

$$\omega_0'' + \delta_1^*\omega_0' + \delta_0^*\omega_0 = 0 \in \bar{V}. \tag{46}$$

Then

$$\delta_1^* = \frac{\Phi'}{\Phi} - \frac{\Phi_*'}{\Phi_*} + \left(\frac{1}{\lambda} + \frac{1}{\lambda-1}\right)\lambda' - \frac{\lambda''}{\lambda'}, \quad \text{and} \quad \delta_0^* = -\frac{\Phi_*'}{\Phi_*}\delta_1^* - \frac{\Phi_*''}{\Phi_*}.$$

Here $\Phi' = \partial\Phi/\partial\rho$ and $\omega_0' = \nabla(D)\omega_0$, etcetera.

Proof: Let

$$\eta := E\frac{\lambda(\lambda-1)}{\lambda'}\omega_0 + \frac{\lambda(\lambda-1)}{\lambda'}\omega_0' \tag{47}$$

be a generator of the unit root part of $\bar{V}$. Write $\nabla(D)\eta = -H\eta$ and $F\eta = G\eta$. It follows from the commutativity of the diagram (41) that $H \equiv G'/G \bmod p$. Therefore Lemma 4.5.1 and the fact that $F\xi = \Phi\xi$ implies that $G = \Phi$ and

$$H \equiv \frac{\Phi'}{\Phi} \pmod{p}.$$

Recall from Section 4.3 that

$$\frac{F}{p}\omega_0 = \frac{1}{\Phi_*}\omega_0 + D\eta,$$

for some $D \in k((\rho))$. Together with (43), this implies that $E \equiv -\Phi_*'/\Phi_* \bmod p$.

One computes that

$$\begin{aligned}
\nabla(\partial/\partial\rho)\eta &= \left[E'\frac{\lambda(\lambda-1)}{\lambda'} + E(2\lambda-1) - E\lambda(\lambda-1)\frac{\lambda''}{(\lambda')^2}\right]\omega_0 + \\
&\quad + \left[E\frac{\lambda(\lambda-1)}{\lambda'} + (2\lambda-1) - \lambda(\lambda-1)\frac{\lambda''}{(\lambda')^2}\right]\omega_0' + \frac{\lambda(\lambda-1)}{\lambda'}\omega_0'' \\
&= -HE\frac{\lambda(\lambda-1)}{\lambda'}\omega_0 - H\frac{\lambda(\lambda-1)}{\lambda'}\omega_0'.
\end{aligned}$$

The last equality follows from the fact that $\nabla(D)\eta = -H\eta$. This implies that

$$\delta_1^* = H + E + \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\lambda' - \frac{\lambda''}{\lambda'} = \frac{\Phi'}{\Phi} - \frac{\Phi'_*}{\Phi_*} + \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\lambda' - \frac{\lambda''}{\lambda'} \tag{48}$$

and

$$\delta_0^* = HE + E' + E\left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\lambda' - E\frac{\lambda''}{\lambda'} \tag{49}$$

$$= -\frac{\Phi'_*}{\Phi_*}\frac{\Phi'}{\Phi} + \left(\frac{\Phi'_*}{\Phi_*}\right)^2 - \frac{\Phi''_*}{\Phi_*} - \frac{\Phi'_*}{\Phi_*}\left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\lambda' + \frac{\Phi'_*}{\Phi_*}\frac{\lambda''}{\lambda'}. \tag{50}$$

□

4.6 The singularities of the bundle $\mathcal{E}$ In this section we show that the F-crystal $\bar{V}$ extends to filtered flat vector bundle $(\mathcal{E}, \nabla)$ on the whole of B_0. We determine the singularities of $(\mathcal{E}, \nabla)$. We show that ω_0 defines a holomorphic section of $(\mathcal{E}, \nabla)$; it is a generator of the filtration $\mathrm{Fil}^1(\mathcal{E}) \subset \mathcal{E}$.

Proposition 4.6.1 *(a) The bundle $\mathcal{E}$ extends to a filtered flat vector bundle on B_0.*

(b) The singularities of $(\mathcal{E}, \nabla)$ are $b \in \Sigma_0$. These are regular singularities.

Proof: We have already defined $\mathcal{E}$ over U_{ord}. To show that $\mathcal{E}$ extends to B_0, we need to show that for $b \in B_0 - U_{\mathrm{ord}}$, there exists an $\mathcal{O}_{B_0,b}$-lattice $\mathcal{E}_b \subset \bar{V}$ with

$$\nabla(D)(\mathcal{E}_b) \subset \mathcal{E}_b. \tag{51}$$

Here t is a local parameter at b and $D = t\partial/\partial t$ if b is a singularity and $D = \partial/\partial t$ otherwise.

Let $b \in B_0 - U_{\mathrm{ord}} \cup \Sigma_0$ and let t be a local parameter at b. We define $\mathcal{E}_b$ as the intersection of $\mathcal{H}_{\mathrm{dR}}(Z/B) \otimes \mathbb{F}_p$ with $\bar{V}$.

Suppose that $b \notin \Sigma_0$. We claim that b is not a singularity of $(\mathcal{E}, \nabla)$. Set $D = \partial/\partial t$. By definition of Σ_0, we know that the curve $\bar{Z}_{0,b}$ is nonsingular. Therefore b is not a singularity of $(\mathcal{H}_{\mathrm{dR}}(Z/B) \otimes \mathbb{F}_p, \nabla)$. It follows that $\nabla(D)$ stabilizes $\mathcal{H}_{\mathrm{dR}}(Z/B) \otimes \mathbb{F}_p$. We have seen that $\bar{V}$ is generated by ω_0 and $\omega_0' = \nabla(\partial/\partial\lambda)\omega_0$ as $k(B_0)$-vector space. Since

$$\nabla(D)(\omega_0, \omega_0') = (\omega_0'\partial\lambda/\partial t, -\delta_0^*\omega_0\partial\lambda/\partial t - \delta_1^*\omega_0'\partial\lambda/\partial t),$$

it follows that $\bar{V}$ is stabilized by $\nabla(\partial/\partial t)$ also. This shows that b is not a singularity of $(\mathcal{E}, \nabla)$. We define a filtration on $\mathcal{E}_b$ by intersecting the Hodge filtration of $\mathcal{H}_{\mathrm{dR}}(Z/B)$ with $\bar{V}$.

Suppose that $b \in \Sigma_0$ and set $D = t\partial/\partial t$. Define $\mathcal{E}_b \subset \bar{V}$ to be the $\mathcal{O}_{B_0,b}$-lattice spanned by ω_0 and $\nabla(t\partial/\partial t)\omega_0$. It follows from Proposition 4.5.2 that $\nabla(D)$ stabilizes $\mathcal{E}_b$. This shows that the $b \in \Sigma_0$ is a regular singularity of $(\mathcal{E}, \nabla)$. We define the Hodge filtration on $\mathcal{E}_b$ as the line bundle generated by ω_0. □

By definition of $\mathcal{E}$, we have that $\mathcal{E}_b$ is generated by ω_0, ω_0' for b sufficiently general. The following lemma extends this partially to $B_0 - \Sigma_0$.

Lemma 4.6.2 *Let $b \in B_0$. Then ω_0 is nonzero as element of $\mathcal{E}_b$.*

Proof: Let t be a local parameter at $b \in B_0$. It is no restriction to suppose that the leading coefficient u_d of $u \in k(B_0)[x]$ is 1.

If $b \in \Sigma_0$, then ω_0 is nonzero as element of $\mathcal{E}_b$, by definition of $\mathcal{E}_b$ (see the proof of Proposition 4.6.1).

Suppose that $b \notin \Sigma_0$ and that b is not a zero of Φ_*. Then $\mathcal{C}\omega_0 = \Phi_*(b)\omega_0 \neq 0$, hence $\omega_0 \in H^0(\bar{Z}_0, \Omega)$ is nonzero.

Now suppose that $b \notin \Sigma_0$ and that b is a zero of Φ_*. Write $\bar{Z}_{0,b}/k(b)$ for the fiber of $\bar{Z}_0$ at $b \in B_0$. Since the Hodge to de Rham spectral sequence degenerates at level 1 ([14]), we may describe the first de Rham cohomology group in characteristic p by

$$H^1_{\mathrm{dR}}(\bar{Z}_{0,b}/k(b)) = \frac{\{\,(\theta_i, f_{ij})_i \,|\, \theta_i - \theta_j = f_{ij}\}}{(\mathrm{d}f_i, f_i - f_j)\}},$$

with respect to a suitable covering $(U_i)_i$ of $\bar{Z}_{0,b}$. Here θ_i (resp. f_{ij}) is a holomorphic differential on U_i (resp. a holomorphic function on $U_i \cap U_j$).

Write

$$\omega_0 = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)} = \left[\frac{z}{x(x-1)(x-\lambda)u_0u}\right]^p G\,\mathrm{d}x,$$

with

$$G = \prod_{i \in \mathbb{B}_0'} (x-\tau_i)^{p-1-a_i} u^{p-2} = \sum_i g_i x^i.$$

Recall that

$$\sum_i g_{pi-1} x^i = \Phi_* u_0^p u^p.$$

Since Φ_* has a zero in b, it follows therefore that ω_0 is exact. Concretely, we have that

$$\omega_0 = \mathrm{d}f, \qquad \text{with } f = \left[\frac{z}{x(x-1)(x-\lambda)u_0u}\right]^p \sum_{i \not\equiv -1 \bmod p} \frac{g_i x^{i+1}}{i+1}\,\mathrm{d}x.$$

We claim that we cannot represent ω_0 as a coboundary $(\mathrm{d}f_i, f_i - f_j)$. Since ω_0 is holomorphic on $\bar{Z}_{0,b}$, it corresponds to the cocycle $(\theta_i, 0)$. Therefore if we could represent it as a coboundary $(\mathrm{d}f_i, f_i - f_j)$, we would have that $f_i = f_j$ is holomorphic as function on $\bar{Z}_{0,b}$. But it is easy to see that f has poles, for instance above $x = 1$. This proves the lemma. □

4.7 The Kodaira–Spencer map In this section we investigate the Kodaira–Spencer map. We show that it is everywhere nonzero, except possibly at those supersingular points $b \in \Sigma_1$ which are ramified in $\pi : B_0 \to \mathbb{P}^1_\lambda$. The reason for this exception is the following. Our argument showing that the Kodaira–Spencer map is nonzero at the ramification points of π relies on the deformation theory of $\boldsymbol{\mu}_p$-torsors [51]. To see what happens at ramification points of π which are supersingular, one would have to study the deformation theory of $\boldsymbol{\alpha}_p$-torsors which appears to be much more complicated. I have not been able to find an example of a special deformation datum such that the accessary parameter cover π is ramified at a supersingular point, so I do not know whether this actually occurs.

Proposition 4.7.1 *Let $b \in B_0$ and t a local parameter of B_0 at b. If $b \in \Sigma_1$, we suppose that b is unramified in $\pi : B_0 \to \mathbb{P}^1_\lambda$. Write $D = t\partial/\partial t$ (resp. $D = \partial/\partial t$) if $b \in \Sigma_0$ (resp. $b \notin \Sigma_0$). Then $(\omega_0, \nabla(D)\omega_0)$ form a basis of $\mathcal{E}_b$.*

Proof: If $b \in \Sigma_0$, the proposition follows immediately from the definition of $\mathcal{E}_b$ in the proof of Proposition 4.6.1.

Suppose that $b \notin \Sigma_0$ is not a ramification point of $\pi : B_0 \to \mathbb{P}^1_\lambda$. Then $\nabla(\partial/\partial t)\omega_0 = \nabla(\partial/\partial\lambda)\omega_0$. It follows therefore from Lemma 4.5.1 that the image of $\nabla(\partial/\partial t)\omega_0$ in $H^1(\bar{Z}_{0,b}, \mathcal{O})_\chi$ is nonzero.

Suppose $b \notin \Sigma_0$ is a ramification point of $\pi : B_0 \to \mathbb{P}^1_\lambda$. Assume that b is not a supersingular point, i.e. $\mathrm{ord}_b(\Phi_*) \not\equiv \bmod p$. After multiplying ω_0 by a nonzero element of $k(B_0)^p$, we may assume that $\mathrm{ord}_b(\Phi_*) = 0$.

Claim 1: Suppose that $\nabla(\partial/\partial t)\omega_0$ is zero at b. Then $\partial\tau_i/\partial t$ is zero at b, for all $i \in \mathbb{B}_{\mathrm{new}}$.

We have that $\nabla(\partial/\partial t)\omega_0 = (\partial\lambda/\partial t)\nabla(\partial/\partial\lambda)\omega_0$. Therefore (45) states that

$$\nabla(\partial/\partial t)\omega_0 = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)u_0u}\left(\frac{(1+a3)u_0u}{x-\lambda}\frac{\partial\lambda}{\partial t} + u\sum_{i\in\mathbb{B}_0\cap\mathbb{B}_{\mathrm{new}}}\frac{a_i}{x-\tau_i}\frac{\partial\tau_i}{\partial t} - 2u_0\frac{\partial u}{\partial t}\right).$$

Since $\pi : B_0 \to \mathbb{P}^1_\lambda$ is ramified at b it follows that $\partial\lambda/\partial t$ is zero at b. The assumption that $\nabla(\partial/\partial t)\omega_0$ is zero at b implies therefore that

$$\sum_{i\in\mathbb{B}_{\mathrm{new}}}\frac{a_i}{x-\tau_i}\frac{\partial\tau_i}{\partial t} = 0$$

at $t=0$. Since $b \notin \Sigma_0$ the τ_i are all distinct at $t=0$. It follows that $\partial\tau_i/\partial t$ is zero at $t=0$, for all $i\in\mathbb{B}_{\mathrm{new}}$. This proves the claim.

REST OF THE PROOF NEEDS TO BE CORRECTED!!!! To deduce the proposition from the claim, we consider the deformation of deformation data, as in [51].

Let $d_{\mathrm{new}} = |\mathbb{B}_{\mathrm{new}}|$. We consider the moduli space of auxiliary covers $\bar{Z}_0$ given by

$$z^{p-1} = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3}\prod_{i\in\mathbb{B}_{\mathrm{new}}}(x-\tau_i)^{a_i}$$

with fixed signature (σ_i). Concretely $\mathcal{D}\subset\mathbb{A}^{d_{\mathrm{new}}+1}$ is the locus of tuples $\lambda, (\tau_i)_{i\in\mathbb{B}_{\mathrm{new}}}$ such that

$$\omega_0 := \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)}$$

is an eigenvector of the Cartier operator with nonzero eigenvalue. In other words, a multiple ω of ω_0 should be logarithmic. Locally, we define a finite cover $\tilde{\mathcal{D}}\to\mathcal{D}$ which parameterizes logarithmic differential forms on the auxiliary curve $\bar{Z}_0$ with fixed signature (σ_i). Since we restrict to the locus of $\mathbb{A}^{d_{\mathrm{new}}+1}$ where a multiple of ω_0 is logarithmic, this cover is étale. The reason why we need to go to a finite cover is that the logarithmic differential $\omega = c^{1/(p-1)}\omega_0$, where c is the eigenvalue of ω_0 under the Cartier operator. Recall from Section **??** that $\mathcal{D}$ is an (affine) curve.

Claim 2: The moduli space $\mathcal{D}$ is smooth. To prove the claim, it suffices to show that $\tilde{\mathcal{D}}$ is smooth, since $\tilde{\mathcal{D}}\to\mathcal{D}$ is étale. This claim is essentially proved in [51] but not stated in this form. Theorem 4.8 of [51] states that the deformation problem of deformation data with fixed signature admits a versal deformation. This versal deformation space corresponds to the completion of $\tilde{\mathcal{D}}$ at the point corresponding to b.

As in [51, Section 4.1], one defines a singular curve Y over a neighborhood of b in $\tilde{\mathcal{D}}$. Remark 4.6.ii of [51] implies that this curve is generically smooth, since the inertia group of the auxiliary cover has order p in our case. This implies that the sheaf $\mathcal{E}xt^1_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)$ has support in isolated points (namely the singularities of Y.) Therefore [51, (43)] implies that

$$\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/k},\mathcal{O}_Y) = 0.$$

This means that the deformation problem is unobstructed, and the claim follows.

The proposition now follows from claims 1 and 2. Namely, since $\mathcal{D}$ is smooth we have that $\mathcal{O}_{B_0,b}\simeq\mathcal{O}_{\mathcal{D},b}$. Therefore t is a local parameter of $\mathcal{D}$ at b. This contradicts claim 1. □

Lemma 3.4.1 implies that Def $(\bar{Y}_{\mathrm{sing}},\mathcal{G}_0)^{\mathrm{loctriv}}\to\mathcal{I}$ is finite and flat, therefore the corresponding deformation spaces have the same dimension.

Let $b\in B_0$ and let t be a local parameter at b.

Lemma 4.7.2 *(a) The line bundle* Fil $\subset\mathcal{E}$ *is generated by* ω_0.

(b) *The degree of* Fil $\subset \mathcal{E}$ *as line bundle on* B_0 *is zero.*

Proof: By definition of the filtration on $\bar{V}$, we have that Fil is generated generically by ω_0. Since ω_0 defines a nonzero element in $\mathcal{E}_b$ for all $b \in B_0$ (Lemma 4.6.2), (a) follows.

This implies that Fil $\subset \mathcal{E}$, regarded as line bundle on B_0, has degree zero. □

The following theorem is an immediate corollary of Proposition 4.7.1.

Theorem 4.7.3 (a) *The Kodaira–Spencer map* $\kappa = \kappa_{\mathcal{E}} : \mathrm{Fil} \to \mathrm{Gr} \otimes \Omega^{\log}_{B_0/k}$ *is nonzero.*

(b) *Suppose that* $\pi : B_0 \to \mathbb{P}^1_\lambda$ *is unramified at the supersingular points. Then the Kodaira–Spencer map* $\kappa = \kappa_{\mathcal{E}} : \mathrm{Fil} \to \mathrm{Gr} \otimes \Omega^{\log}_{B_0/k}$ *is an isomorphism.*

Proof: To prove the theorem, we compute the degree of $\mathrm{Gr} = \mathcal{E}/\mathrm{Fil}$ as line bundle on B_0. The differential $\theta := \nabla(\partial/\partial\lambda)\omega_0$ is a rational section of Gr.

Let e_b be the ramification index of b in $\pi : B_0 \to \mathbb{P}^1_\lambda$. Let $t = (\lambda - \pi(b))^{1/e_b}$ (resp. $t = \lambda^{-1/e_b}$) be a local parameter at $b \in B_0$, depending on whether $\pi(b) \neq \infty$ (resp. $\pi(b) = \infty$). Put $D = D_b = t\partial/\partial t$ (resp. $D = D_b = \partial/\partial t$) if $b \in \Sigma_0$ (resp. $b \notin \Sigma_0$). Since $\nabla(D)\omega_0 = D(\lambda)\theta$ is nonzero at b by Proposition 4.7.1, it follows that

$$\mathrm{ord}_b(\theta) = \begin{cases} -e_b & \text{if } b \in \Sigma_0 \text{ and } \pi(b) \neq \infty, \\ e_b & \text{if } \pi(b) = \infty, \\ 1 - e_b & \text{if } b \notin \Sigma_0. \end{cases}$$

Here we use the assumption that $\pi : B_0 \to \mathbb{P}^1_\lambda$ is unramified at the supersingular points.

Write $R = \sum_b (e_b - 1)$. Then

$$\deg(\theta) = -\sum_{b \in \Sigma_0} (-e_b) + 2 \sum_{\pi(b)=\infty} e_b + \sum_{b \notin \Sigma_0} (1 - e_b) = -R - s + 2\deg(\pi) = -2g(B_0) + 2 - s.$$

Together with Lemma 4.7.2, this implies that

$$\deg(\mathrm{Gr}) - \deg(\mathrm{Fil}) = -2g(B_0) + 2 - s = -\deg(\Omega^{\log}_{B_0/k}). \tag{52}$$

Therefore κ is an isomorphism. □

Let $D = t\partial/\partial t$ if b is a singularity of $(\bar{V}, \nabla)$ and $D = \partial/\partial t$ otherwise. Then the Kodaira–Spencer map at b may be computed as

$$\kappa(D) : \mathrm{Fil} \to \bar{V} \xrightarrow{\nabla(D)} \bar{V} \to \bar{V}/\mathrm{Fil}(\bar{V}), \qquad \omega_0 \mapsto \nabla(D)\omega_0.$$

Corollary 4.7.4 *Suppose that the cover* $\pi : B_0 \to \mathbb{P}^1_\lambda$ *is unramified at the supersingular points. Let* $b \in B_0 - \Sigma_0$ *and let* e_b *the ramification index of* b *in* $\pi : B_0 \to \mathbb{P}^1_\lambda$. *Then*

$$\mathrm{ord}_b(\Phi_*) \equiv 0, 1 \bmod p \quad \textit{and} \quad \mathrm{ord}_b(\Phi) \equiv \mathrm{ord}(\Phi_*) + e_b - 1 \bmod p.$$

Proof: Let $b \notin \Sigma_0$ and let t be a local parameter at b. Write $D = \partial/\partial t$. Write $\alpha_* := \mathrm{ord}_b(\Phi_*)$ and $\alpha := \mathrm{ord}_b(\Phi)$. Taking $\rho = t$ in Proposition 4.5.2, one finds that

$$\nabla(D)^2\omega_0 + \delta_1^*\nabla(D)\omega_0 + \delta_0^*\omega = 0,$$

with

$$\begin{aligned} \delta_1^* &= \frac{\alpha - \alpha_* - e_b + 1}{t} + \text{ higher order terms,} \\ \delta_0^* &= \frac{\alpha_*(-\alpha + e_b)}{t^2} + \text{ higher order terms.} \end{aligned}$$

Theorem 4.7.3 (or also Proposition 4.7.1) states that $\omega_0, \nabla(D)\omega_0$ form a basis of $\mathcal{E}_b$. Therefore δ_1^* and δ_0^* are regular in b. This implies that $\alpha \equiv \alpha_* + e_b - 1$ and $\alpha \equiv e_b \bmod p$ or $\alpha_* \equiv 0 \bmod p$. □

In Section 4.9 we give a variant of Corollary 4.7.4 which applies without the assumption on the supersingular points.

4.8 The p-curvature In this section we show that the p-curvature $\Psi_{\mathcal{E}}$ of $\mathcal{E}$ is nonzero. We also express the order n_b of a spike $b \in B_0$ modulo p in terms of the ramification index and the order at b of Φ_* and Φ.

Proposition 4.8.1 *The p-curvature $\Psi_{\mathcal{E}}$ of $\mathcal{E}$ is nilpotent and nonzero.*

Proof: It is well know that $\Psi_{\mathcal{E}}$ is nilpotent (resp. nonzero) if and only if the p-curvature of the differential operator $L := (\partial/\partial\rho)^2 + \delta_1^*(\partial/\partial\rho) + \delta_0^*$ is nilpotent (resp. nonzero) ([17, Appendix]).

The nilpotence of Ψ_L follows from the fact that $\Phi_*'' + \delta_1^*\Phi_*' + \delta_0^*\Phi_* = 0$. Namely, as explained in Section 4.1, it suffices to show that

$$w' + \delta_1^* w = 0 \tag{53}$$

has a solution. Proposition 4.5.2 implies that $w = \Phi_*(\partial\lambda/\partial\rho)/\Phi\lambda(\lambda-1)$ is a solution to (53).

Suppose that $\Psi_{\mathcal{E}} = 0$. Then [24, Theorem 5.1] implies that $\bar{V} = \mathcal{E} \otimes k(B_0)$ is generated by its horizontal sections. More precisely, this result states that

$$\bar{V} \xrightarrow{\sim} (F^*(\bar{V}))^{\nabla_{\rm can}},$$

where $\nabla_{\rm can}$ is the canonical connection on $F^*\bar{V}$ whose horizontal sections consist precisely of the pth powers, i.e. the sections of the form $e^{\otimes p}$, where e is a section of $\bar{V}$.

Choose $\theta_1, \theta_2 \in \bar{V}$ horizontal elements which generate $\bar{V}$. It follows from the previous discussion that we may choose θ_1, θ_2 to be eigenvectors of F. We know that one of the eigenvalues of F is zero, say $F\theta_1 = 0$. Since the kernel of F on $\bar{V}$ is $\mathrm{Fil}(\bar{V})$, it follows that $\theta_1 = v\omega_0$, for some $v \in k(B_0)$. But this contradicts the assumption that θ_1 is horizontal, since $\nabla(\partial/\partial\lambda)$ does not fix $\mathrm{Fil}(\bar{V})$ (Lemma 4.5.1). This proves that $\Psi_{\mathcal{E}}$ is nonzero. □

Theorem 4.8.2 *The bundle $(\mathcal{E}, \nabla)$ is pseudo elliptic.*

Proof: Theorem 4.7.3 implies that the Kodaira–Spencer map $\kappa : \mathrm{Fil} \to \mathrm{Gr} \otimes \Omega^{\log}_{B_0/k}$ is nonzero. The statement that the p-curvature $\Psi_{\mathcal{E}}$ of $\mathcal{E}$ is nilpotent but nonzero is proved in Proposition 4.8.1. □

As in Section 4.1, we let $\mathcal{M} \subset \mathcal{E}$ be the kernel of the p-curvature $\Psi_{\mathcal{E}}(D^{\otimes p})$, where D is some rational section of $\tau^{\log}_{B_0/k}$. Recall from Section 4.1 that $\mathcal{M}$ is stabilized by $\nabla(D)$. Proposition 4.8.1 implies that $\mathcal{M}$ is uniquely characterized by this property. Moreover, the restriction of $\mathcal{M}$ to the ordinary locus $U_{\rm ord}$ corresponds to the unit root part of the F-crystal $\bar{V}$ (Section 1.3). Recall that we have an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

of flat vector bundles, where the p-curvature of $\mathcal{M}$ and $\mathcal{L}$ is zero. The p-curvature of $\mathcal{E}$ can be regarded as a nonzero, horizontal homomorphism

$$\Psi_{\mathcal{E}} : \mathcal{T} \longrightarrow \mathcal{M} \otimes \mathcal{L}^{-1}.$$

In particular, for any vector field D we may regard $\Psi_{\mathcal{E}}(D^{\otimes p})$ as a horizontal section of $\mathcal{M} \otimes \mathcal{L}^{-1}$.

Let ρ be a parameter on B_0. For example, one could choose $\rho = \lambda$. In what follows $'$ denotes derivation with respect to ρ.

Let us compute the p-curvature in the singularities $b \in \Sigma_0$. Let t be a local parameter at the point b_i, and set $D_i := t\partial/\partial t$. It is easy to see that $D_i^p = D_i$. Therefore,

$$\Psi_{b_i} := \Psi_{\mathcal{E}}(D_i^{\otimes p})|_{b_i} = \mu_i^p - \mu_i.$$

Hence if $\mathcal{E}$ has a logarithmic singularity at b_i with exponent α_i then

$$\Psi_{b_i} \sim \begin{pmatrix} \alpha_i^p - \alpha_i & -1 \\ 0 & \alpha_i^p - \alpha_i \end{pmatrix},$$

which is nonzero at b. If $\mathcal{E}$ has a toric singularity at b with exponents α_i, β_i then

$$\Psi_{b_i} \sim \begin{pmatrix} \alpha_i^p - \alpha_i & 0 \\ 0 & \beta_i^p - \beta_i \end{pmatrix}.$$

Since $\Psi_{\mathcal{E}}$ is nilpotent, it follows that the local exponents α_i, β_i are elements of $\mathbb{F}_p$. Therefore Ψ_{b_i} has a zero in the toric singularities. In the terminology of Section 4.1: the toric singularities are spikes of $\mathcal{E}$. From now on we suppose that α_i is the local exponent of the subbundle $\mathcal{M}$ of $\mathcal{E}$ at $b = b_i$.

Remark 4.8.3 If $\pi : B_0 \to \mathbb{P}^1_\lambda$ is unramified at the supersingular points, then $(\mathcal{E}, \nabla)$ is an indigenous bundle, as defined in [10]. Namely Theorem 4.7.3 states that under this assumption the Kodaira–Spencer map is an isomorphism. To show that $\mathcal{E}$ is indigenous, it remains to check that the monodromy at the marked points is nonzero. If $b \in \Sigma_0$ has logarithmic monodromy, this holds by definition.

Suppose that $b = b_i \in \Sigma_0$ has toric monodromy. Recall from the proof of Lemma 4.8.4 that $\alpha_i \neq \beta_i$. This obviously implies that the monodromy at b_i is nonzero.

The next lemma computes the order n_b of the spikes. The same result in a somewhat different set-up is proved in [10, Prop. 2.2].

Lemma 4.8.4 *(a) Let $b = b_i$ be a logarithmic singularity. Then $n_b = 0$.*

(b) Let $b = b_i$ be a toric singularity. Then $n_b \equiv \beta_i - \alpha_i \not\equiv 0 \bmod p$.

(c) If b is not a singularity, then $n_b \equiv 0 \bmod p$.

Proof: Part (a) follows from the discussion preceeding the lemma. Suppose that $b = b_i$ has toric monodromy. Then $\mathcal{M} \otimes \mathcal{L}^{-1}$ has a regular singularity with local exponent $\alpha_i - \beta_i$. Let D be some derivation, and regard $\Psi_{\mathcal{E}}(D^{\otimes p})$ as horizontal section of $\mathcal{M} \otimes \mathcal{L}^{-1}$. One checks that this implies that $\Psi_{\mathcal{E}}$ has a zero whose order is congruent to $\beta_i - \alpha_i \bmod p$. Suppose that $\alpha_i \equiv \beta_i \bmod p$. Then ω_0 is an eigenvalue of the monodromy operator μ_i. But this contradicts the fact that $(\omega_0, \nabla(t\partial/\partial t)\omega_0)$ form a basis of $\mathcal{E}_b$. (See the proof of Proposition 4.6.1.)

Suppose that $b \notin \Sigma_0$. Then $\mathcal{M} \otimes \mathcal{L}^{-1}$ does not have a singularity at b. Hence the same argument as above implies that $n_b \equiv 0 \bmod p$. □

Let us express the local exponents α_i, β_i of a singularity $b = b_i$ in terms of $\alpha := \mathrm{ord}_b(\Phi)$, and $\alpha_* := \mathrm{ord}_b(\Phi_*)$, and the ramification index e of b in $\pi : B_0 \to \mathbb{P}^1_\lambda$. Let t be a local parameter at b_i.

Suppose that $\pi(b) \neq 0, 1, \infty$. Then

$$\begin{aligned} \delta_1^* &= \frac{\alpha - \alpha_* - e + 1}{t} + \text{ higher order terms}, \\ \delta_0^* &= \frac{\alpha_*(-\alpha + e)}{t^2} + \text{ higher order terms}. \end{aligned}$$

Therefore the indicial equation is $X^2+(\alpha-\alpha_*-e)X+\alpha_*(-\alpha+e)=(X+\alpha-e)(X-\alpha_*)$. Since

$$\hat{\eta}=-\Phi\frac{\partial\Phi_*/\partial t}{\Phi_*}\frac{\lambda(\lambda-1)}{\partial\lambda/\partial t}\omega_0+\Phi\frac{\lambda(\lambda-1)}{\partial\lambda/\partial t}\nabla(\partial/\partial t)\omega_0$$

is horizontal, we conclude as in the proof of Lemma 4.8.4 that $e-\alpha$ is the local exponent corresponding of $\mathcal{M}$, i.e. $\alpha_i=e-\alpha$ and $\beta_i=\alpha_*$.

Similarly, if $\pi(b)=0,1$, we have that the indicial equation is $(X-\alpha_*)(X+\alpha)$. We have that $\alpha_i=-\alpha$ and $\beta_i=\alpha_*$.

If $\pi(b)=\infty$, we find that $\alpha_i=e-\alpha$ and $\beta_i=\alpha_*$. This proves the following lemma.

Lemma 4.8.5 *Let $b=b_i\in\Sigma_0$. Then*

$$n_{b_i}\equiv\begin{cases}\alpha_*+\alpha \bmod p & \text{if } \pi(b_i)=0,1,\\ \alpha_*+\alpha-e_{b_i} \bmod p & \text{if } \pi(b_i)\neq 0,1\end{cases}$$

4.9 The supersingular points In this section we investigate what happens at the supersingular points, without assuming that the supersingular points are unramified in the cover $\pi: B_0\to\mathbb{P}^1_\lambda$. The following proposition is a generalization of Corollary 4.7.4.

Proposition 4.9.1 *Let $b\in B_0-\Sigma_0$ be a supersingular point, and write e_b for the ramification index of b in $\pi: B_0\to\mathbb{P}^1_\lambda$. Then*

(a) $\mathrm{ord}_b(\Phi_*)\equiv\mathrm{ord}_b(\Phi)\equiv e_b \bmod p$,

(b) the Kodaira–Spencer map has a zero of order $\gamma_b\equiv e_b-1 \bmod p$,

(c) after tensoring with $\overline{k(b)}$, the group scheme $\mathcal{G}_b$ is isomorphic to $E[p]$, where $E/\mathbb{F}_{p^2}$ is a supersingular elliptic curve.

Proof: Let b be a supersingular point and write e_b for its ramification index. Let t be a local parameter of $b\in B_0$ and put $D=\partial/\partial t$. Lemma 4.6.2 implies that ω_0 is nonzero as element of $\mathcal{E}_b$. Therefore $\nabla(D)\omega_0$ is regular at b. Define $\gamma_b=\mathrm{ord}_b(\nabla(D)\omega_0)$. Then $\theta:=t^{-\gamma_b}\nabla(D)\omega_0$ is nonzero in $\mathcal{E}_b$ and not contained in $\mathrm{Fil}_b\subset\mathcal{E}_b$. It follows that $\nabla(D)\theta$ is regular at b. Therefore (ω_0,θ) form a basis of $\mathcal{E}_b$.

Define $\alpha_*=\mathrm{ord}_b(\Phi_*)$ and $\alpha=\mathrm{ord}_b(\Phi)$. One computes that

$$\nabla(D)\theta = \omega_0\left(-\frac{\alpha_*(-\alpha+e_b)}{t^{2+\gamma_b}}+\text{higher order terms}\right)+ \tag{54}$$

$$-\theta\left(\frac{-\gamma_b+\alpha-\alpha_*-e_b+1}{t}+\text{ higher order terms}\right). \tag{55}$$

Since $\alpha_*\not\equiv 0 \bmod p$ by definition of supersingularity, it follows that $\alpha\equiv e_b \bmod p$ and $\gamma_b\equiv\alpha_*-1 \bmod p$.

Next we consider the differential equation corresponding to the dual F-crystal $\bar{V}^*$. Recall that $\omega_{0,*}=\mathrm{d}x/z\in H^0(\bar{Z}_0,\Omega)_{\chi^{-1}}$ is the basis vector dual to $\xi=z/x\in H^1(\bar{Z}_0,\mathcal{O})_\chi$ under Serre duality (up to an $\mathbb{F}_p$-constant which we may ignore.) Using this, one computes as in Section 4.5 that $\omega_{0,*}$ satisfies the differential equation

$$\nabla^*(D)^2\omega_{0,*}+\delta_1\nabla^*(D)\omega_{0,*}+\delta_0\omega_{0,*}, \tag{56}$$

where

$$\delta_1 = \frac{D(\Phi)}{\Phi}-\frac{D(\Phi_*)}{\Phi_*}+\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)D(\lambda)-\frac{D^2(\lambda)}{D(\lambda)},$$

$$\delta_0 = -\frac{D(\Phi_*)}{\Phi_*}\frac{D(\Phi)}{\Phi}+\left(\frac{D(\Phi)}{\Phi}\right)^2-\frac{D^2(\Phi)}{D(\Phi)}-\frac{D(\Phi)}{\Phi}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)D(\lambda)+\frac{D(\Phi)}{\Phi}+\frac{D^2(\lambda)}{D(\lambda)}.$$

Note that δ_0, δ_1 are obtained from δ_0^* and δ_1^* by reversing the role of Φ_* and Φ. The differential equation (56) is the dual differential equation as defined in Section 4.1, expressed with respect to the basis vector $\lambda(\lambda-1)e_2^*$, where (e_1^*, e_2^*) is the basis dual to $(e_1 = \omega_0, e_2 = \nabla(D)\omega_0)$. The reason for the appearance of the factor $\lambda(\lambda-1)$ is explained by Lemma 4.5.1. Recall that Φ is an expansion coefficient of $\omega_{0,*}$. It can be easily checked directly that Φ is a solution of (9). This observation can be used to give an alternative computation of the dual differential equation.

The argument of Lemma 4.6.2 also implies that $\omega_{0,*}$ is nonzero as element of $\mathcal{H}_{\mathrm{dR}}(\bar{Z}_0)_{\chi^{-1}}$. Define $\gamma_b^* = \mathrm{ord}_b(\nabla^*(D)\omega_{0,*})$, and let $\theta_* = t^{-\gamma_b^*}\nabla^*(D)\omega_{0,*}$. Proposition 4.4.1 together with the assumption that b is a supersingular point implies that $\alpha = \mathrm{ord}_b(\Phi) \not\equiv 0 \bmod p$. Applying the argument of (54) to the dual differential equation (56), we obtain that $\alpha_* \equiv e_b \bmod p$ and $\gamma_b^* \equiv \alpha - 1 \bmod p$. We conclude that $\alpha \equiv \alpha_* \equiv e_b \bmod p$. This proves (a) and (b).

To finish the proof, we determine the structure of the group scheme $\mathcal{G}_b$. Recall that η defined in (47) is a rational section of $\mathcal{M}$ which satisfies $F\varphi^*(\eta) = \Phi\eta$. Using that $F\varphi^*(\omega_0) = 0$, one computes that

$$F\varphi^*\theta = \frac{D(\lambda)}{\lambda(\lambda-1)t^{\gamma_b}}\Phi\eta = -\frac{D(\Phi_*)\Phi}{\Phi_* t^{\gamma_b}}\omega_0 + \Phi\theta.$$

Since $\alpha > 0$ and $\alpha - 1 - \gamma_b \equiv e_b - \alpha_* \equiv 0 \bmod p$, it follows that $F\varphi^*(\theta) = (\text{ unit })\omega_0$ at b. A similar argument applied to θ_* shows that $F\varphi^*(\theta_*) = (\text{ unit })\omega_{0,*}$ at b. Dualizing and using that $\lambda(\lambda-1)$ is nonzero at b since $b \notin \Sigma_0$, we find that $V\theta = (\text{ unit })\omega_0$. Part (c) follows. □

Remark 4.9.2 Let b be supersingular and write J_b for the Jacobian of $\bar{Z}_{0,b}$. Recall from Corollary 4.4.2 that $J_b[p]_\chi \simeq (\mathbb{Z}/p)^a \times \Lambda(b)$, with $\Lambda(b)$ an indecomposable local-local group scheme which surjects onto $\mathcal{G}_b$. It follows easily from the description of local-local group schemes ([29]) together with Proposition 4.9.1.(c) that such a surjection only exists if the rank of $\Lambda(b)$ is p^2, i.e. if $J_b[p]_\chi \simeq (\mathbb{Z}/p)^a \times \mathcal{G}(b)$.

Lemma 4.9.3 *(a) The degree of $\mathcal{L}$ as line bundle on B_0 is equal to $\sum_{b\in\Sigma_1}\gamma_b$.*

(b) The degree of $\mathcal{M}$ as line bundle on B_0 is $\deg(\mathcal{L}) + \sum_b n_b - p(2(g(B_0) - 2 + s))$.

Proof: Let $b \in B_0$ and t be a local parameter at b. As usual, we write $D = t\partial/\partial t$ (resp. $D = \partial/\partial t$) depending on whether $b \in \Sigma_0$ or not. We write $[\omega_0]$ (resp. $[\nabla(D)\omega_0]$) for the (rational) section of $\mathcal{L}_b$ induced by ω_0 (resp. $\nabla(D)\omega_0$).

First suppose that b is not a supersingular point. Proposition 4.7.1 implies that

$$\min(\mathrm{ord}_b[\omega_0], \mathrm{ord}_b[\nabla(D)\omega_0]) = 0.$$

Since $-D(\Phi_*)\omega_0 + \Phi_*\nabla(D)\omega_0$ is a rational section of $\mathcal{M}$ (cf. (47)), it follows that

$$[\nabla(D)\omega_0] = \frac{D(\Phi_*)}{\Phi_*}[\omega_0].$$

Hence

$$\mathrm{ord}_b[\omega_0] = 0.$$

Now let b be a supersingular point and write e_b for the ramification index of b in $\pi : B_0 \to \mathbb{P}^1_\lambda$. Then Proposition 4.9.1 implies that $\mathrm{ord}_b([\nabla(D)\omega_0]) = \gamma_b \equiv e_b - 1 \bmod p$. As above, it follows that $\mathrm{ord}_b([\omega_0]) = \gamma_b + 1 \equiv e_b \mod p$. This proves (a).

For (b), choose a derivation D and consider $\Psi_{\mathcal{E}} : \mathcal{T} = (\tau_{B_0/k}^{\log})^{\otimes p} \to \mathcal{M} \otimes \mathcal{L}^{-1}$. The definition of the order n_b of a spike implies that

$$\deg(\mathcal{M}) - \deg(\mathcal{L}) = \sum_b n_b + \deg\mathcal{T}.$$

This finishes the proof since $\deg(\mathcal{T}) = -p(2(g(B_0) - 2 + s))$. □

Lemma 4.8.5 computes the degree of $\mathcal{M}$ modulo p.

Define ψ to be the natural map

$$\psi : \mathcal{M} \longrightarrow \mathcal{E} \longrightarrow (\mathcal{E}/\,\mathrm{Fil}).$$

The map ψ is called the *Hasse invariant.*

The following lemma shows that the supersingular points are the zeros of ψ. Therefore it makes sense to call both ψ and Φ_* the Hasse invariant.

Proposition 4.9.4 *(a) A point $b \in B_0 - \Sigma_0$ is supersingular if and only if ψ has a zero at b.*

(b) The map ψ has a zero of order γ_b in the supersingular points and is regular elsewhere.

Proof: Let t be a local parameter of B_0 at $b \notin \Sigma_0$ and let $e = e_b$ be the ramification index of b in $\pi : B_0 \to \mathbb{P}^1_\lambda$. Let $D = \partial/\partial t$.

Let $\epsilon = 0$ if b is not supersingular and $\epsilon = \gamma_b$ otherwise. Then for η as in (47) we have

$$\eta_t := \frac{\partial \lambda}{\partial t} t^\epsilon \eta = -\frac{D(\Phi_*)}{\Phi_*}\lambda(\lambda-1)t^\epsilon\omega_0 + \lambda(\lambda-1)t^\epsilon\nabla(D)\omega_0$$

generates $\mathcal{M}_b$. (This follows from Proposition 4.9.1 if b is supersingular and Corollary 4.7.4 otherwise.) Therefore $\psi(\eta_t) = [\lambda(\lambda-1)t^\epsilon\nabla(D)\omega_0] \in \mathrm{Gr}_b$, and ψ has a zero of order ϵ.

The proof for $b \in \Sigma_0$ is similar. □

Corollary 4.9.5 *We have that*

$$\sum_b n_b = (p-1)(2g(B_0) - 2 + s).$$

Proof: Proposition 4.9.4 implies that $\deg(\psi) = \sum_{b\in\Sigma_1}\gamma_b$ is equal to $\deg(\mathcal{M}) - \deg(\mathrm{Gr})$. Therefore the corollary follows from Theorem 4.7.3 and Lemma 4.9.3. □

Example 4.9.6 We illustrate the results of this section in an easy example. Let $p \geq 7$ be a prime and take $\mathbf{a} = (1, p-4, p-4, 1)$. It follows that $d = p - 1 - (a_0 + a_1 + a_2 + a_3)/2 = 2$. To find a Fuchsian deformation datum of type $\mathbf{a}$, we need to find a polynomial solution of degree 2 of $P_0u'' + P_1u' + P_2u = 0$, for some choice of the accessary parameter β. Here $P_0 = x(x-1)(x-\lambda)$, $P_1 = -7x^2 + (6\lambda + 1)x - 3\lambda$ and $P_2 = 6x + \beta$. The recursion (31) easily implies that β should satisfy

$$(\beta + 6\lambda)(10\beta\lambda + 30\lambda + \beta^2 + \beta) = 0.$$

Suppose that $\beta + 6\lambda = 0$, for simplicity. (The other case can be analyzed analogously, compare to Section 4.11.) Then the unique monic solution of degree 2 of $P_0u'' + P_1u' + P_2u = 0$ is $u = x^2 - 2\lambda x + \lambda$. It follows that

$$\Phi_* = 6(\lambda - 1/2) \qquad \text{and} \qquad \Phi = 4\lambda(\lambda-1)(\lambda-1/2).$$

This implies that

$$\delta_1^* = \frac{2}{\lambda} + \frac{2}{\lambda-1}, \qquad \text{and} \qquad \delta_0^* = -2\frac{2\lambda-1}{\lambda(\lambda-1)(\lambda-1/2)} = \frac{-4}{\lambda(\lambda-1)}.$$

Therefore $\mathcal{E}$ has no logarithmic singularities and three toric singularities $\lambda = 0, 1, \infty$ with local exponents $0, -1; 0, -1; 4, -1$ respectively.

4.10 The deformation datum corresponding to $\mathcal{E}$ In [10] it is shown that one can associate to a deformation datum to a pseudo-elliptic bundle. We make this construction explicit in our case.

We now recall from [10, Section 3.3] the construction of the deformation datum corresponding to $\mathcal{E}$. For this construction we do not need to assume that $\pi : B_0 \to \mathbb{P}^1_\lambda$ is unramified at the supersingular points.

For simplicity, we suppose that there exists a parameter ρ on B_0 which has a pole in one point ∞ with $\pi(\infty) = \infty$, and that $\mathrm{ord}_b(\partial\rho/\partial t) = 0$ for $b \neq \infty$. Let $D = \partial/\partial\rho$. Then $D^p = 0$. As before, we write

$$\eta = -\frac{D(\Phi_*)}{\Phi_*}\frac{\lambda(\lambda-1)}{D(\lambda)}\omega_0 + \frac{\lambda(\lambda-1)}{D(\lambda)}\nabla(D)\omega_0.$$

It is a rational section of $\mathcal{M}$ which satisfies

$$\nabla(D)\eta = -\frac{D(\Phi)}{\Phi}\eta, \qquad \text{and} \qquad F\eta = \Phi\eta.$$

Therefore $\Phi\eta$ is horizontal.

Define $W_1 \in k(B_0)$ to be a rational function such that $W_1^p\Phi\eta$ is a section of $\mathcal{M}_b$ for all $b \neq \infty$. We may choose W_1 to be minimal in the sense that $0 \leq \mathrm{ord}_b(W_1) < p$ for all $b \in \Sigma_0 - \{\infty\}$. We call $\hat{\eta} := W_1^p\Phi\eta$ the *minimal generator* of $\mathcal{M}$ (with respect to the choice of ρ).

Define $e_1 := W_1^p\omega_0/\Phi_*$. One computes that

$$\nabla(D)e_1 = -\frac{D(\Phi_*)}{\Phi_*^2}W_1^p\omega_0 + \frac{1}{\Phi_*}W_1^p\nabla(D)\omega_0 = \frac{D(\lambda)}{\Phi\Phi_*\lambda(\lambda-1)}\hat{\eta} =: v\hat{\eta}. \tag{57}$$

Since $\hat{\eta}$ is horizontal, it follows that

$$\Psi_{\mathcal{E}}(D^{\otimes p})e_1 = D^{p-1}(v)\hat{\eta}.$$

Alternatively,

$$\Psi_{\mathcal{E}}(D^{\otimes p}) = D^{p-1}(v)\hat{\eta} \otimes [e_1]^{-1},$$

as (horizontal) section of $\mathcal{M} \otimes \mathcal{L}^{-1}$.

Write $v = \sum_i v_i\rho^i$. Then $D^{p-1}v = -\sum_i v_{pi-1}\rho^{pi}$ is a pth-power, say

$$D^{p-1}v =: -W^p. \tag{58}$$

Replacing e_1 by $e_0 := e_1/W^p = \omega_0 W_1^p/(\Phi_* W^p)$, we find therefore that $\nabla(D)e_0 = (v/W^p)\hat{\eta}$ and

$$\Psi_{\mathcal{E}}(D^{\otimes p})e_0 = -\hat{\eta}.$$

The section e_0 is the analog in our situation of what is called the *canonical section* in [10]. It is well-defined up to multiplication by an element of $k^\times$. We refer to [10] for more details.

The deformation datum corresponding to $\mathcal{E}$ is now defined as follows. Let C_0 be a connected component of the nonsingular projective curve with generic equation

$$y^{p-1} = \frac{W^p}{v}.$$

Note that C_0 is a cyclic cover of B_0 of order dividing $p-1$. Put $\theta := y\,\mathrm{d}\rho$.

Lemma 4.10.1 *The differential θ on C_0 is logarithmic.*

Proof: Put $D = \partial/\partial\rho$. We have that $\theta = y\,\mathrm{d}\rho = (y/W)^p v\,\mathrm{d}\rho$. Applying the Cartier operator, we find

$$\mathcal{C}\theta = \mathcal{C}\left(\frac{y}{W}\right)^p v\,\mathrm{d}\rho = -\frac{y}{W}(D^{p-1}v)^{1/p}\,\mathrm{d}\rho = y\,\mathrm{d}\rho,$$

since $D^{p-1}(v) = -W^p$. This shows that θ is logarithmic. □

The following lemma is proved in [10]. It expresses the signature (Section 2.2) of the deformation datum in terms of the orders of the spikes.

Lemma 4.10.2 *Let $b \in B_0$ and write σ_b for the ramification invariant of the deformation datum (C_0, θ), as defined in Section 2.2. Then*

$$n_b = \begin{cases} 0 & \text{if } b \text{ is supersingular and unramified in } \pi : B_0 \to \mathbb{P}^1_\lambda, \\ (p-1)\sigma_b & \text{if } b \in \Sigma_0, \\ (p-1)(\sigma_b - 1) & \text{otherwise.} \end{cases}$$

Corollary 4.9.5 is just the Riemann–Roch Theorem applied to the differential form θ. It gives a formula for the sum of the σ_i. Lemma 4.8.5 now computes the signature of the deformation datum modulo p. Moreover we know that $\sigma_i \geq 0$. Unfortunately, this information does not determine the σ_i completely.

In Section 5.3 we will interprete the differential θ as a Swan conductor of a certain cover $\varpi : \mathbb{H} \to \mathbb{P}^1_\lambda$. This is the Galois closure of $\mathcal{H} \to \mathbb{P}^1_\lambda$, where $\mathcal{H}$ is a certain Hurwitz space parameterizing G-Galois covers of $\mathbb{P}^1$ branched at four points defined over a number field. The over ϖ is branched at three points. In case p strictly divides the order of the Galois group of ϖ, we may use the results of [40] and [50]. It is shown in [40, Proposition 3.3.5] that $\sigma_i > 1$ for $i \in \mathbb{B}_{\mathrm{new}}$. The vanishing cyclic formula (Corollary 4.9.5) implies that there are three primitive critical points. Moreover, $0 \leq \sigma_i \leq 1$ for $i \in \mathbb{B}_{\mathrm{prim}}$ and $1 < \sigma_i \leq 2$ for $i \in \mathbb{B}_{\mathrm{new}}$. Together with the formula for the σ_i modulo p, this is enough to determine the signature. (Compare to Section 6.2.) It would be interesting to know whether Raynaud's estimate for the ramification invariant of the new critical points also holds if p^2 divides the order of the Galois group of ϖ.

4.11 An example In this section we give a more involved example of a pseudo-elliptic bundle. We focus here on the role of the accessary parameter. Let $p \geq 7$ be a prime number and consider $\mathbf{a} := (1, p-2, p-6, 1)$. It follows that $d = 2(p-1) - (a_0 + a_1 + a_2 + a_3) = 2$.

We need to find a polynomial solution $u = u_2 x^2 + u_1 x + u_0$ of the differential equation $P_0 u'' + P_1 u' + P_2 u = 0$ for some choice of the accessary parameter β, where

$$P_0 = x(x-1)(x-\lambda), \qquad P_1 = -4x^2 + (6\lambda - 1)x - \lambda, \qquad P_2 = 6x + \beta.$$

The recursion (31) implies that

$$u_1 = \frac{\beta}{\lambda} u_0, \qquad -6\lambda(\beta+1) + \beta - \beta^2 = 0, \qquad u_2(4\lambda - 10\lambda^2 - \lambda\beta) - 2\beta u_0 = 0.$$

Therefore

$$\lambda = \frac{-\beta(\beta-1)}{6(\beta+1)}.$$

Choosing $u_0 = (\beta - 1)(\beta^2 + 2\beta + 6)$, we find that

$$u = -18(\beta+1)^2 x^2 - 6(\beta+1)(\beta^2 + 2\beta + 6)x + (\beta-1)(\beta^2 + 2\beta + 6).$$

This choice is made in such a way that u does not have denominators and its coefficients are relatively prime. We denote by $\pi : B_0 \to \mathbb{P}^1_\lambda$ the cover defined by $\lambda = -\beta(\beta-1)/6(\beta+1)$. It is ramified at $\beta^2 + 2\beta - 1 = 0$. We choose β as parameter on B_0.

We first determine the set Σ_0 of points where the curve $\bar{Z}_0$ given by $z^{p-1} = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3} u^2$ is singular. It follows from Proposition 4.2.2, Lemma 3.2.1 and the fact that u has degree 2 that $\bar{Z}_0$ is singular if and only $\lambda = 0, 1, \infty$ or $u(0) = 0$. One computes that this corresponds to the set $\beta = 0, 1, -2, -3, -1, \infty$ and $\beta^2 + 2\beta + 6 = 0$. Therefore Σ_0 has cardinality 8.

It follows from the explicit expression of Φ_* and Φ given in Section 4.4 that

$$\Phi_* = \frac{(\beta+3)^2}{(\beta+1)^5}, \qquad \Phi = \frac{(\beta-1)^2(\beta^2+2\beta+6)^2(\beta^2+2\beta-1)}{\beta+1}.$$

Note that Φ has a zero at points with $\beta^2 + 2\beta - 1 = 0$, but Φ_* does not. This shows that the converse of Proposition 4.4.1 does not hold. Note that there are no supersingular points.

We write $\nabla(\partial/\partial\beta)^2\omega_0 = -\delta_0^*\omega_0 - \delta_1^*\nabla(\partial/\partial\beta)\omega_0$. Using Proposition 4.5.2, we find that

$$\delta_1^* = \frac{2\,(6\,\beta^6 + 40\,\beta^5 + 105\,\beta^4 + 156\,\beta^3 + 89\,\beta^2 - 54\,\beta - 18)}{(\beta^2 + 5\,\beta + 6)\,\beta\,(\beta^2 + 2\,\beta + 6)\,(\beta^2 - 1)}$$

and

$$\delta_0^* = \frac{2\,(12\,\beta^6 + 92\,\beta^5 + 261\,\beta^4 + 408\,\beta^3 + 403\,\beta^2 + 198\,\beta - 78)}{(\beta^2 + 5\,\beta + 6)\,\beta\,(\beta - 1)\,(\beta^2 + 2\,\beta + 6)\,(\beta + 1)^2}.$$

The local exponents at $\beta = 0, 1, -1, \infty, -2, -3$ are $0, 0; 0, -2; 2, -5; 3, 8; 0, 0; 0, 2$, respectively. At the roots of $\beta^2 + 2\beta + 6$ the local exponents are $0, -1$.

One checks that indeed

$$\frac{\partial^2\Phi_*}{\partial\beta^2} + \delta_1^*\frac{\partial\Phi_*}{\partial\beta} + \delta_0^*\Phi_* = 0.$$

Let v be as in (57), i.e.

$$v = \frac{\partial\lambda/\partial\beta}{\Phi\Phi_*\lambda(\lambda - 1)} = \frac{-6(\beta + 1)^6}{(\beta - 1)^3(\beta^2 + 2\beta + 6)^2(\beta + 3)^3\beta(\beta + 2)}.$$

One computes that $\mathrm{Res}_1 v = \mathrm{Res}_0 v = -\mathrm{Res}_{-2} v = -\mathrm{Res}_{-3} v = 1/324$. Therefore

$$D^{p-1}v = \frac{(\beta^2 + 2\beta - 1)^p}{54[\beta(\beta - 1)(\beta + 2)(\beta + 3)]^p} = -\frac{(\partial\lambda/\partial\beta)^p(\beta + 1)^{2p}}{324\lambda^p(\lambda - 1)^p} = -W^p.$$

The corresponding deformation datum is

$$y^{p-1} = \frac{W^p}{v} = \frac{(\beta^2 + 2\beta - 1)^p(\beta^2 + 2\beta + 6)^2}{\beta^{p-1}(\beta - 1)^{p-3}(\beta + 2)^{p-1}(\beta + 3)^{p-3}(\beta + 1)^6}, \qquad \theta = y\,\mathrm{d}\beta.$$

The signature of this deformation datum is therefore

b	0	1	-2	-3	-1	∞	$\beta^2 + 2\beta + 6 = 0$	$\beta^2 + 2\beta - 1 = 0$
σ_b	0	$\frac{2}{p-1}$	0	$\frac{2}{p-1}$	$\frac{p-7}{p-1}$	$\frac{p-5}{p-1}$	$\frac{p+1}{p-1}$	$\frac{2p-1}{p-1}$.

4.12 Families of elliptic curves In this section we define elliptic bundles; this is a (slightly simplified) mod p-version of elliptic crystals as defined by Ogus [38, Definition 1.1]. Let R be a discrete valuation ring with residue field k an algebraically closed field of characteristic p and fraction field K of characteristic zero.

Definition 4.12.1 Let B_0/k be a complete nonsingular curve, and let $\mathcal{E}/B_0$ be a pseudo-elliptic bundle. We say that $\mathcal{E}$ is an *elliptic bundle* if there exists horizontal isomorphism

$$\mathrm{tr} : \bigwedge^2 \mathcal{E} \longrightarrow \mathcal{O}_{B_0}$$

which is compatible with the Frobenius morphism in the following sense. Write $\langle\cdot,\cdot\rangle : \mathcal{E}^2 \to \mathcal{O}_{b_0}$ for the alternating bilinear form corresponding to tr. Then the compatibility with the Frobenius morphism on $\mathcal{E}$ amounts to

$$\langle F\varphi^* x, F\varphi^* y\rangle = p\varphi^*\langle x, y\rangle. \tag{59}$$

Let $\mathcal{E}/B_0$ be the pseudo-elliptic bundle associated to some special deformation datum. Let $b \in B_0 - \Sigma_0 \cup \Sigma_1$, and write $\mathcal{E}_b$ for the fiber at b. The existence of a horizontal isomorphism $\mathrm{tr} : \wedge^2\mathcal{E} \to \mathcal{O}_{B_0}$ corresponds to the choice of a horizontal vector. Choose a derivation D and write

$$\omega_0' = \nabla(D)\omega_0, \qquad \eta = -\frac{D(\Phi_*)}{\Phi_*}\lambda(\lambda-1)\omega_0 + \lambda(\lambda-1)\omega_0'.$$

It follows from the results of Section 4.3 and 4.5 that

$$\frac{F}{p}\varphi^*\omega_0 = \frac{1}{\Phi_*}\omega_0 + D_1\eta, \quad \text{and} \quad F\varphi^*\omega_0' = \frac{\Phi}{\lambda(\lambda-1)}\eta.$$

This implies that

$$\langle \frac{F}{p}\varphi^*\omega_0, F\varphi^*\omega_0' \rangle = \frac{\Phi}{\Phi_*\lambda(\lambda-1)}\langle \omega_0, \eta\rangle = \frac{\Phi}{\Phi_*}\langle \omega_0, \omega_0'\rangle.$$

Therefore the condition of Definition 4.12.1 is satisfied if and only if $\Phi = \Phi_*$.

Lemma 4.12.2 *Let $b \in B_0 - \Sigma_0 \cup \Sigma_1$. Then $\mathcal{E}_b$ admits a trace map if and only if b is unramified in $\pi : B_0 \to \mathbb{P}^1_\lambda$.*

Proof: Since $b \in B_0 - \Sigma_0 \cup \Sigma_1$, the curve $\bar{Z}_{0,b}$ is smooth, and b is not supersingular. The lemma follows easily for the explicit expressions for e_0 and $\hat{\eta}$, together with the unicity statements of Section 4.10 and the expression for the order of the zeros of Φ and Φ_* (Corollary 4.7.4). $\square$

The lemma is easy to understand in terms of the group scheme $\mathcal{G}_b$ (Section 4.4). The trace map corresponds to a duality on $\mathcal{E}_b$ which corresponds to Cartier duality on $\mathcal{G}_b$. But we have seen that if $b \in B_0 - \Sigma_0 \cup \Sigma_1$ is a ramification point of $\pi : B_0 \to \mathbb{P}^1_\lambda$, then $\mathcal{G}_b \simeq \mathbb{Z}/p \times \boldsymbol{\alpha}_p$. Therefore $\mathcal{G}_b$ is not isomorphic to its Cartier dual; it follows that a trace map as in Definition 4.12.1 does not exist.

Suppose that $B_R/\operatorname{Spec}(R)$ is a complete, nonsingular curve, and $g_R : E_R \to B_R$ a semistable family of elliptic curves. We write $g_0 : E_0 \to B_0$ for the reduction modulo p, and assume that it is not isotrivial.

The Gauß–Manin connection

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(E_R/R) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(E_R/R) \otimes \Omega^{\log}_{B/R}$$

makes $\mathcal{H} := \mathcal{H}^1_{\mathrm{dR}}(E_R/R)$ into a flat vector bundle with logarithmic singularities [24]. Denote by $\mathrm{Fil}^1(\mathcal{H})$ the filtration induced by the Hodge filtration.

Write $E_0 = E_R \otimes_R k$, and σ_0 for the set of points $b \in B_0 := B \otimes_R k$ for which the elliptic curve $E_{0,b}$ is singular.

Lemma 4.12.3 *The bundle $\mathcal{H} := \mathcal{H}^1_{\mathrm{dR}}(E_0)$ is an elliptic bundle.*

Proof: Since we assumed that $g_0 : E_0 \to B_0$ is not isotrivial, the fiber of E_0 above the generic fiber of B_0 is a smooth ordinary elliptic curve. It is well known that this implies that the Kodaira–Spencer map of $\mathcal{H}$ is nontrivial. The statement that the p-curvature $\Psi_{\mathcal{H}}$ is nilpotent is shown in [24]. The statement that the p-curvature is nonzero follows again from the assumption that E_0 is generically ordinary, by using that the Frobenius morphism vanishes on $\mathrm{Fil}^1(\mathcal{H}) \subset \mathcal{H}$.

Let $\mathrm{tr} : \wedge^2\mathcal{H} \to \mathcal{O}_{B_0}$ be the natural map induced by Serre duality which identifies $\mathrm{Fil}^1(\mathcal{H}) = \mathcal{H}^0(E_0, \Omega)$ with the dual of $\mathrm{Gr}^0(\mathcal{H}) = \mathcal{H}/\mathrm{Fil}^1(\mathcal{H}) = \mathcal{H}^1(E_0, \mathcal{O})$. Write $\bar{V} = \mathcal{H} \otimes k(B_0)$, and

$$\langle \cdot, \cdot \rangle : \mathrm{Fil}^1(\bar{V}) \times \mathrm{Gr}^0(\bar{V}) \longrightarrow k(B_0)$$

for the corresponding alternating pairing. It is well known that this pairing satisfies $\langle Vx, y\rangle = \langle x, Fy\rangle$, where $V : \bar{V} \to \bar{V}$ is the Verschiebung. We claim that this is a trace map as in Definition

4.12.1. Let $\omega_0 \in \Gamma(B_0, \mathrm{Fil}^1(\mathcal{H})) = H^0(E_0, \Omega^1)$ corresponds to the invariant differential form on E_0, and let $\xi \in H^1(E_0, \mathcal{O})$ be the dual basis vector with respect Serre duality. Write $F\xi = \Phi\xi$. As in Section 4.3 it follows that we may lift ξ to a rational section η of $\mathcal{H}$ such that $F\eta = \Phi\eta$. Note that this terminology is somewhat misleading; the notation is not completely consistent with the notation in the rest of Section 4. Since we are in characteristic p, we have $VF = 0$. This implies that $V\omega_0 = \Phi\omega_0$. This shows that $\mathcal{H}$ is an elliptic bundle. □

Write $\nabla(D)^2\omega_0 + p_1\nabla(D)\omega_0 + p_2\omega_0 = 0 \in \mathcal{H}$ for the Picard–Fuchs differential equation. As in Section 4.3 it follows from $V\omega_0 = \Phi\omega_0$ that Φ satisfies the same differential equation, i.e. $D^2(\Phi) + p_1 D(\Phi) + p_2\Phi = 0$. From this one easily deduces as in Section 4.5 that

$$\eta = -\frac{D(\Phi)}{\Phi} w\omega_0 + w\nabla(D)\omega_0,$$

where w satisfies $p_1 = \mathrm{d}w/w$, i.e. w is essentially a solution of the Wronskian equation, cf. Section 4.1. Moreover, one checks that

$$\nabla(D)\frac{\omega_0}{\Phi} = \frac{1}{w\Phi}\eta.$$

As in Section 4.10 this implies that

$$\Psi_{\mathcal{E}}(D^{\otimes p})(\frac{\omega_0}{\Phi}\omega_0, \Phi\eta) = \begin{pmatrix} 0 & 0 \\ D^{p-1}\dfrac{1}{w\Phi^2} & 0 \end{pmatrix} (\frac{\omega_0}{\Phi}\omega_0, \Phi\eta)$$

We finish this section with a concrete example. We formulate this here in the more classical terms of families of elliptic curves, but it is clearly equivalent to the formulation in terms of deformation data as we did before.

Example 4.12.4 Consider one of the families of elliptic curves over a projective line with four singular fibers found by Beauville [2]. Picard–Fuchs differential equations of some of these families have been computed by Stienstra and Beukers [44] in characteristic zero. They also consider the differential equation in mixed characteristic zero, and relate the unit root eigenvalue of Frobenius to solutions of the Picard–Fuchs differential equation, similar in spirit to the discussion in Section 1. For similar computations on the family we consider here see [46].

Let $p > 2$ be a prime, and $B = \mathbb{P}^1_{\mathbb{Z}_p}$. The family of semistable elliptic curves over B we consider is given by

$$E_t : \qquad (x+y)(xy - z^2) = \frac{xyz}{t}.$$

This family is the universal elliptic curve with a $\Gamma := \Gamma_0(8) \cap \Gamma_1(4)$-level structure. The elliptic curve E_t is singular if and only if $t \in \Sigma_0 := \{0, \infty, \pm i/4\}$. One checks that the modular curve B of level Γ admits a degree two cover $\pi : B \to X_0(8)$ ramified at $t = 0, \infty$. Denote by Σ_0' the image of Σ_0 on $X_0(8)$. Since π is Galois, this set has cardinality three. Clearly, what we compute below is only a small illustration on all what can be said here. For example, the relation to K3-surfaces is not touched upon. We refer to [44].

It is computed in [46] that the Picard–Fuchs differential equation of $\mathcal{H} := \mathcal{H}^1_{\mathrm{dR}}(E/k(B_0))$ is given by

$$L := t(16t^2 + 1)(\partial/\partial t)^2 + (48t^2 + 1)(\partial/\partial t) + 16t.$$

We may choose an isomorphism $X_0(8) \simeq \mathbb{P}^1_s$ such that the Picard–Fuchs differential equation corresponding to $X_0(8)$ is Gauß' hypergeometric differential equation

$$L' := s(s-1)(\partial/\partial s)^2 + (2s-1)(\partial/\partial s) + \frac{1}{4}.$$

In particular, $\Sigma_0' \simeq \{0,1,\infty\}$. Moreover, $\pi : B \to X_0(8)$ is given by $s = \pi(t) = -16t^2$ and L is the pull-back of L' via π. This gives an alternative way of computing the Picard–Fuchs differential equation in characteristic p.

We now consider the differential equation L in characteristic p Let

$$\Phi' = \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i}^2 s^i \in \mathbb{F}_p[s]$$

be the classical Hasse invariant. It satisfies $L(\Phi') = 0$. Therefore

$$\Phi(t) := \Phi'(-16s^2) \tag{60}$$

satisfies $L(\Phi) = 0$. Since $\deg_t(\Phi) = p-1$, this implies that Φ is the Hasse invariant of E, in the notation of the proof of Lemma 4.12.3 (cf. Section 3).

Let $w = t(t^2 + 1/16)$ be the minimal solution of $\partial w/\partial t = (48t^2+1)/(16t^3+t) = \delta_1^*$. Put $v = 1/(w\Phi^2)$. As in Section 3, it follows from the differential equation that the residue of v at a zero of Φ (a supersingular point) is zero. Write $\{b_1, b_2\} = \pi^{-1}(1) = \{\pm i/4\}$. Then

$$\mathrm{Res}_{t=0} v = -\frac{1}{\Phi(0)^2}, \quad \mathrm{Res}_{t=b_1} v = \mathrm{Res}_{t=b_2} v = \frac{1}{b_1(b_1-b_2)\Phi(b_1)^2}$$

It follows from (60) and the well-known fact that $\Phi'(0) = 1$ and $\Phi'(1) = (-1)^{(p-1)/4}$ that $\Phi(0) = 1$ and $\Phi(b_i)^2 = 1$. Therefore

$$\mathrm{Res}_{t=0} v = 16, \qquad \mathrm{Res}_{t=b_1} v = \mathrm{Res}_{t=b_2} v = -8.$$

Writing $D = \partial/\partial s$, we find that

$$D^{p-1} v = -\frac{16}{t^p} + \frac{8}{(t-b1)^p} + \frac{8}{(t-b2)^p} = -\frac{1}{t^p(t^2+1/16)^p} =: -W^p.$$

This describes the deformation datum corresponding to the elliptic bundle, as in Section 4.10. We find

$$y^{(p-1)/2} = \Phi^2, \qquad \theta = \frac{y}{t(t^2+1/16)}\,\mathrm{d}t.$$

5 The Swan conductor of a Hurwitz curve

Let $f : Y \to \mathbb{P}^1$ be a G-Galois cover branched at four ordered points in characteristic zero, and $\mathcal{H}$ the component of the Hurwitz space of G-Galois covers such that f corresponds to a point of $\mathcal{H}$. The goal of this section is to relate the reduction to characteristic p of f with the reduction of the natural map $\pi : \mathcal{H} \to \mathbb{P}^1_\lambda$. Assume that f has special bad reduction to characteristic p, and let $(\mathcal{E}, \nabla)$ be the corresponding pseudo-elliptic bundle. The main result of this section interpretates $(\mathcal{E}, \nabla)$ as a differential Swan conductor in the sense of Kato associated to the Galois closure of π (Theorem 5.3.2). In Section 5.1 we review Kato's definition of the Swan conductor. The proof of our result relies on the determination of the minimal field over which the stable reduction $\bar{f}$ of f may be lifted to characteristic zero (Proposition 5.2.3).

5.1 Review of Kato's Swan conductors We define the Swan conductor of a finite Galois extension L/K of complete discrete valued fields whose residue field extension is purely inseparable, following Kato [22]. In case the degree of L/K is p, this Swan conductor is just the deformation datum of the residue field extension (Example 5.1.5).

Let K be a complete discrete valuation field, with residue class field k of characteristic $p > 0$. We write $\mathcal{O}_K$ for the valuation ring of K and $\mathfrak{m}_K$ for its maximal ideal. We denote by v_K the normalized valuation, with $v_K(K^\times) = \mathbb{Z}$. Given an element $x \in \mathcal{O}_K$, we write $\bar{x} \in k$ for its residue class. We make the following assumption on the residue field k.

Assumption 5.1.1 The field k has an absolute p-basis of length 1.

Equivalently, the k-vector space of absolute differentials Ω_k of k has dimension 1. A unit $x \in \mathcal{O}_K^\times$ such that $\mathrm{d}\bar{x}$ is a basis of Ω_k is called a *generator* of K. Another equivalent formulation of Assumption 5.1.1 is that

$$[k : k^{p^n}] = p^n,$$

for all $n \geq 0$. See [31, p. 201ff].

We define the group S_K as the group of units of the k-algebra

$$\bigoplus_{i,j \in \mathbb{Z}} \mathfrak{m}_K^i / \mathfrak{m}_K^{i+1} \otimes \Omega_k^{\otimes^j}.$$

For an element $x \in K^\times$, let $[x]$ denote the corresponding element of $\mathfrak{m}_K^i/\mathfrak{m}_K^{i+1} \subset S_K$ (with $i := v_K(x)$). Similarly, for an element $\omega \in \Omega_k^{\otimes j}$, we write $[\omega]$ for the corresponding element of S_K. The group law for S_K is written *additively*. Thus, if we fix a generator x of K and a prime element π_K, then every element of S_K can be written in the form

$$[f\,(\mathrm{d}\bar{x})^{\otimes^i}] + n \cdot [\pi_K],$$

for unique integers i, n and a unique element $f \in k$. In other word, the choice of x and π_K yields an isomorphism $S_K \cong k^\times \oplus \mathbb{Z}^2$.

Let K be as before, and L/K a finite Galois extension, which satisfies the following condition.

Assumption 5.1.2 The extension of residue class fields l/k is purely inseparable, of degree

$$[l : k] = [L : K] = p^n,$$

and generated by one element, i.e. $l = k(\bar{x})$.

This assumption corresponds to *Case II* in Kato's paper [22]. An element $x \in \mathcal{O}_L^\times$ whose residue class $\bar{x}$ generates the extension l/k is called a *generator* of L/K. Such an element is automatically a generator of the field L (in the sense we gave this term above).

Note that if K satisfies Assumption 5.1.1 and L/K satisfies Assumption 5.1.2, then L satisfies Assumption 5.1.1 as well. We have natural injections

$$\mathfrak{m}_K/\mathfrak{m}_K^2 \hookrightarrow \mathfrak{m}_L/\mathfrak{m}_L^2, \qquad \Omega_k \hookrightarrow \Omega_l^{\otimes^{p^n}}.$$

The last map sends $f\,\mathrm{d}\bar{x}^{p^n} \in \Omega_k$ to $f\,(\mathrm{d}\bar{x})^{p^n} \in \Omega_l^{\otimes p^n}$, where $\bar{x}$ is an arbitrary generator of the extension l/k. Therefore, we obtain a natural injection

$$S_K \hookrightarrow S_L.$$

One checks easily that the quotient group S_L/S_K is killed by $p^n = [L : K]$.

Fix a generator x of L/K. For $\sigma \in \mathrm{Gal}(L/K)$, $\sigma \neq 1$, we define

$$s_{L/K}(\sigma) := [\mathrm{d}\bar{x}] - [x - \sigma(x)] \in S_L.$$

One easily checks that this definition is independent of the choice of x. We also set

$$s_{L/K}(1) := -\sum_{\sigma \neq 1} s_{L/K}(\sigma).$$

The element $s_{L/K}(1) \in S_L$ is also called the *different* of L/K, and is denoted by $\mathfrak{D}_{L/K}$. The different is the Swan conductor of the augmentation ideal.

Let H be a normal subgroup of $\mathrm{Gal}(L/K)$, and $M := L^H$. Then for all $\tau \in \mathrm{Gal}(M/K)$, $\tau \neq 1$, we have

$$s_{M/K}(\tau) = \sum_{\sigma \mapsto \tau} s_{L/K}(\sigma), \tag{61}$$

see [22, Proposition 1.9]. In particular, the right hand side of (61) lies in $S_M \subset S_L$. One easily deduces from (61) the transitivity of the different, i.e. the formula

$$\mathfrak{D}_{L/K} = \mathfrak{D}_{L/M} + \mathfrak{D}_{M/K}. \tag{62}$$

Let L/K be a Galois extension satisfying the Assumptions 5.1.1 and 5.1.2. Set $G := \mathrm{Gal}(L/K)$. Note that G is a p-group. Let $\tilde{\mathbb{Z}}$ denote the ring of algebraic integers. We fix a pth root of unity $\zeta \in \tilde{\mathbb{Z}}$, and define

$$\epsilon(\zeta) := \sum_{a \in \mathbb{F}_p^\times} [a] \otimes \zeta^a \in S_K \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}.$$

Note that $\epsilon(\zeta^a) = [a] + \epsilon(\zeta)$.

Definition 5.1.3 Let $\chi : G \to \tilde{\mathbb{Z}}$ be a virtual character. The *Swan conductor* of χ (with respect to $\zeta \in \tilde{\mathbb{Z}}$) is the element

$$\mathrm{sw}_{L/K}(\chi) := \sum_{\sigma \in G} s_{L/K} \otimes \chi(\sigma) + \chi(1) \cdot \epsilon(\zeta) \in S_L \otimes \tilde{\mathbb{Z}}.$$

Proposition 5.1.4 *(a)* $\mathrm{sw}_{L/K}(\chi) \in S_K$.

(b) Let H be a subgroup of G, $M := L^H$, χ a virtual character of H and $\tilde{\chi}$ the induced virtual character on G. Then

$$\mathrm{sw}_{L/K}(\tilde{\chi}) = |G/H| \cdot (\mathrm{sw}_{L/M}(\chi) + \chi(1) \cdot \mathfrak{D}_{M/K}).$$

(c) Let H be a normal subgroup of G, $M := L^H$, χ a virtual character of G/H and χ' the restriction of χ to G. Then

$$\mathrm{sw}_{L/K}(\chi') = \mathrm{sw}_{M/K}(\chi).$$

Note that (a),(b) and (c) are analogies of well known properties of the classical Swan conductor, see e.g. [41]. Here (b) and (c) are more or less formal consequences of (61) and (62), whereas (a) corresponds to the Hasse-Arf Theorem and is quite deep. For a proof, see [22, Proposition 3.3 and Theorem 3.4].

By Proposition 5.1.4.(a), we can write

$$\mathrm{sw}_{L/K}(\chi) = \delta(\chi) \cdot [\,\pi_K\,] - [\,\omega(\chi)\,],$$

with $\delta(\chi) \in \mathbb{Z}$ and $\omega(\chi) \in \Omega_k^{\otimes n}$, $n \in \mathbb{Z}$. Following [18], we call $\delta(\chi)$ the *depth* of χ and $\omega(\chi)$ the *differential Swan conductor* of χ. The integer $\delta(\chi)$ is called the discriminant in [39]. Note that $\omega(\chi)$ depends implicitly on the choice of the prime element π_L.

Example 5.1.5 Suppose G is cyclic of order p. Suppose, moreover, that K contains a primitive pth root of unity ζ. In particular, K has characteristic 0. (We do not distinguish $\zeta \in K$ from $\zeta \in \tilde{\mathbb{Z}}$.) By Kummer theory, $L = K(y)$, with $x := y^p \in \mathcal{O}_K^\times$, and we have a generator σ of G such that $\sigma(y) = \zeta y$.

We distinguish two cases. In the first case, we suppose that $\bar{x} \notin k^p$. Then y is a generator of the extension L/K, and we have

$$s_{L/K}(\sigma^a) = [\,\frac{\mathrm{d}\bar{y}}{\bar{y}}\,] - [\,\lambda\,] - [\,a\,],$$

for all $a \in \mathbb{F}_p^\times$, and with $\lambda := \zeta - 1$. Now if $\chi : G \to \tilde{\mathbb{Z}}$ is a character with $\chi(\sigma) = \zeta^b$, then

$$\begin{aligned}
\mathrm{sw}_{L/K}(\chi) &= \left(\sum_{a \in \mathbb{F}_p^\times} \zeta^{ab} - 1 \right) \cdot \left([\, \frac{\mathrm{d}\bar{y}}{\bar{y}} \,] - [\, \lambda \,] \right) - \epsilon(\zeta^b) + \epsilon(\zeta) \\
&= -p \cdot ([\, \frac{\mathrm{d}\bar{y}}{\bar{y}} \,] - [\, \lambda \,]) - [\, b \,] \\
&= [\, \lambda^p \,] - [\, b \frac{\mathrm{d}\bar{x}}{\bar{x}} \,].
\end{aligned}$$

Hence, the depth of χ is

$$\delta(\chi) = \frac{p \cdot e_K}{p-1},$$

where $e_K := v_K(p)$ is the absolute ramification index of K. Furthermore, if we choose a suitable root of λ as prime element π_L, then the differential Swan conductor is

$$\omega(\chi) = b \cdot \frac{\mathrm{d}\bar{x}}{\bar{x}}.$$

For the second case, we suppose that $\bar{x}$ is a pth power in k. Then one can show that $x = z^p(1 + \pi_K^{pn} u)$, with $z, u \in \mathcal{O}_K^\times$, $\bar{u} \notin k^p$ and $0 < n < e_K/(p-1)$, see e.g. [18]. Write $y = z(1 + \pi_L^n w)$. Then $\bar{w}^p = \bar{u}$, hence w is a generator of L/K. Therefore, we get

$$s_{L/K}(\sigma^a) = [\, \mathrm{d}\bar{w} \,] - [\, \lambda \pi_L^{-n} \,] - [\, a \,].$$

A similar calculation as above yields

$$\begin{aligned}
\mathrm{sw}_{L/K}(\chi) &= -p \cdot ([\, \mathrm{d}\bar{w} \,] - [\, \lambda \pi_L^{-n} \,]) - [\, b \,] \\
&= [\, \lambda^p \pi_L^{-pn} \,] - [\, b \mathrm{d}\bar{u} \,].
\end{aligned}$$

Hence, the depth of χ is

$$c(\chi) = \frac{p \cdot e_K}{p-1} - pn,$$

and the differential Swan conductor is

$$\omega(\chi) = b \cdot \mathrm{d}\bar{u}.$$

5.2 The auxiliary cover In this section we recall Raynaud's construction of the auxiliary cover ([40]). This will be used in the next section in describing the Swan conductor of a Hurwitz curve. The following notation will be fixed in the rest of this section.

Let R be a complete discrete valuation ring of mixed characteristic p, let L be its fraction field and ℓ be its residue field. Let $f : Y \to \mathbb{P}^1$ be a G-Galois cover branched at four points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$, defined over L. After extending L, we may suppose that the stable reduction of f is defined over L. We suppose that

(a) p strictly divides the order of G,

(b) the ramification indices of f are prime-to-p,

(c) λ is transcendental over $\mathbb{Q}_p$,

(d) f has special reduction (Section 2.2).

As usual, we denote the stable model of $f : Y \to \mathbb{P}^1$ by $f_R : Y_R \to X_R$ and the stable reduction by $\bar{f} : \bar{Y} \to \bar{X}$. We choose an irreducible component $\bar{Y}_0$ of $\bar{Y}$ above the original component $\bar{X}_0$. Write $G_0 \subset G$ (resp. I_0) for the decomposition group (resp. inertia group) of $\bar{Y}_0$ and let $\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0$ be the corresponding Galois cover with Galois group $H_0 := G_0/I_0$. We denote by $\chi : H_0 \to \mathbb{F}_p^\times$ the character describing the action by conjugation of H_0 on I_0, as in Section 2.2. We fix a lift $H_0 \subset G_0$. Let ω be the differential form corresponding to $\bar{Y}_0 \to \bar{Z}_0$. Then $(\bar{g}_0, \omega)$ is a special deformation datum. Assumption (c) implies that ω is a logarithmic differential form (Proposition 2.3.3). We assume furthermore that

(e) Φ is nonzero.

(Compare to Assumption 4.2.1.)

We start by recalling the definition of the auxiliary cover of f from [40, Section 3.2]. As in Section 2.2, we write $\mathbb{B} = \mathbb{B}_{\rm prim} \cup \mathbb{B}_{\rm new}$ for the set of tails of $\bar{X}$. Let $\bar{X}_i$ be the irreducible component corresponding to $i \in \mathbb{B}$. Recall that $\bar{X}_i$ intersects the original component $\bar{X}_0$ in a unique point τ_i. For $i \in \mathbb{B}_{\rm prim}$, we denote by $\bar{x}_i$ the specialization of the branch point x_i of $f : Y \to \mathbb{P}^1$ to $\bar{X}_i$. For $i \in \mathbb{B}_{\rm new}$, we choose a ℓ-rational point $\bar{x}_i \in \bar{X}_i - \{\tau_i\}$. We also choose a lift x_i of $\bar{x}_i$ to a L-rational point of X.

It is shown in [40, Section 3.2] that there exists a G_0-Galois cover $f^{\rm aux} : Y^{\rm aux} \to X$ over L with $Y^{\rm aux}$ smooth which is branches at $(x_i)_{i \in \mathbb{B}}$. The cover $f^{\rm aux} : Y^{\rm aux} \to X$ has special stable reduction over L, and its stable reduction gives rise to the deformation datum $(\bar{g}_0, \omega)$. Informally, the stable reduction $\bar{f}^{\rm aux} : \bar{Y}^{\rm aux} \to \bar{X}$ looks as follows. The restriction of $\bar{f}^{\rm aux}$ to the original component $\bar{X}_0$ is $\bar{Y}_0 \to \bar{X}_0$. Let $\bar{Y}_i$ be an irreducible component of $\bar{Y}^{\rm aux}$ above the tail $\bar{X}_i$. We fix an intersection point η_i of $\bar{Y}_i$ with $\bar{Y}_0$. The restriction of $\bar{f}^{\rm aux}$ to $\bar{Y}_i$ is a separable Galois cover $\bar{f}_i : \bar{Y}_i \to \bar{X}_i$ branched only at τ_i and x_i whose Galois group is the decomposition group in $\bar{f} : \bar{Y} \to \bar{X}$ of η_i. The ramification of $\bar{f}_i$ above τ_i is "the same" as the ramification of τ_i in the restriction of $\bar{f}$ to $\bar{X}_i$. Since we assume that $\bar{f}$ has special reduction, this just means that both covers have the same ramification invariant σ_i ([50, Lemma 2.12]).

In Section 4.10, we associated to $(\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0, \omega)$ another deformation datum (C_0, θ). We call this the *Hurwitz deformation datum*. (The relevant Hurwitz space is defined in Section 5.3.) Recall that it lives on a certain cover $B_0 \to \mathbb{P}^1_\lambda$ in characteristic p. The deformation datum of the Hurwitz space defines therefore maps

$$D_0 \to C_0 \to B_0 \to \mathbb{P}^1_\lambda,$$

where $B_0 \to \mathbb{P}^1_\lambda$ is the cover defined by the accessary parameters (Section 3.4), $C_0 \to B_0$ is a cyclic cover of order dividing $p-1$ (Section 4.10) and $D_0 \to C_0$ is the $\boldsymbol{\mu}_p$-torsor corresponding to the logarithmic differential form θ on C_0. We denote the function fields of these curves over $\bar{\mathbb{F}}_p$ by

$$k(D_0) \supset k(C_0) \supset k(B_0) \supset k(\lambda).$$

For the application we have in mind, we are only interested in the wild ramification of L, therefore it is no restriction to replace $k(C_0)$ (resp. $k(D_0)$) by the separable closure $k(\lambda)^{\rm sep}$ (resp. $k(D_0)^{\rm sep}$) in some fixed algebraic closure of $k(\lambda)$.

We denote by $K(\lambda)$ the function field of $\mathbb{P}^1_\lambda$ over $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$. Let v_0 be the valuation of $K(\lambda)$, and write $\hat{K}(\lambda)$ for the completion of $K(\lambda)$ with respect to v_0. We denote by $\hat{K}_1(\lambda)$ the maximal tamely ramified extention of $\hat{K}(\lambda)$; its residue field is $k(\lambda)^{\rm sep}$.

Lemma 5.2.1 *The G_0-Galois cover $f^{\rm aux} : Y^{\rm aux} \to X$ may be defined over an extention $\hat{K}_2(\lambda)$ of $\hat{K}_1(\lambda)$ of degree p. The residue field of $\hat{K}_2(\lambda)$ is $k(D_0)^{\rm sep}$.*

Proof: Write Z_R for the quotient of $Y_R^{\rm aux}$ by the normal subgroup P of G_0 and write $\bar{Z}$ for its special fiber. We denote by $\bar{Z}_i$ the image of $\bar{Y}_i$ in $\bar{Z}$. For $i \in \mathbb{B}$, the induced cover $\bar{Z}_i \to \bar{X}_i$ is a

Galois cover of degree prime to p branched at at most two points, it follows that the genus of $\bar{Z}_i$ is zero. In other words, the cover $g : Z \to X$ has good reduction if we forget the markings; its reduction is just $\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0$. It follows that Z may by defined over $\hat{K}_1(\lambda)$.

It remains to consider the minimal extention of $\hat{K}_1(\lambda)$ over which we can define $Y^{\rm aux} \to Z$. Let $Y_{R,*}^{\rm aux}$ be the (singular) curve obtained by contracting the tails of $Y_R^{\rm aux}$. Its special fiber has cusps as singularities ([50, Section 3]), and admits a $\boldsymbol{\mu}_p$-torsor $\bar{Y}_*^{\rm aux} \to \bar{Z}_0$ (loc. cit., the curve $\bar{Y}_*^{\rm aux}$ was denoted by $\bar{Y}_{\rm sing}$ in Section 3.4). This $\boldsymbol{\mu}_p$-torsor corresponds to the logarithmic differential form ω on $\bar{Z}_0$.

The differential form ω on $\bar{Z}_0$ corresponds to a line bundle $\bar{\mathcal{L}} \in J(\bar{Z}_0)[p](k(C_0))_\chi$. Recall from Section 3.4 that the set of lifts of $\bar{\mathcal{L}}$ to an element of $J(Z)[p]_\chi$ is a torsor under $\boldsymbol{\mu}_p$. This torsor defines an extension $\hat{K}_2(\lambda)/\hat{K}_1(\lambda)$ of degree p. By construction, the corresponding residue field extention is exactly $k(D_0)^{\rm sep}/k(\lambda)^{\rm sep}$. It follows that $f^{\rm aux} : Y^{\rm aux} \to X$ may be defined over $\hat{K}_1(\lambda)$. This proves the lemma. □

Lemma 5.2.2 *The auxiliary cover $f^{\rm aux} : Y^{\rm aux} \to X$ has stable reduction over $\hat{K}_2(\lambda)$.*

Proof: The proof of this proposition follows from [40, Section 4.2], together with a more precise variant found in [50, Lemma 2.17].

Let $L^{\rm st}$ be the minimal extension of $\hat{K}_2(\lambda)$ over which $f^{\rm aux} : Y^{\rm aux} \to X$ acquires stable reduction. Let $\Gamma_w = {\rm Gal}(L^{\rm st}, \hat{K}_2(\lambda))$. Since $\hat{K}_1$ is the maximal tamely ramified extension of $\hat{K}(\lambda)$, it follows that Γ_w is a p-group. Steps 2 and 3 of the proof of [40, Proposition 4.2.4] directly carry over to our situation, and show that Γ_w acts trivially on $\bar{Y}_0$ and the primitive tails $(\bar{X}_i)_{i \in \mathbb{B}_{\rm prim}}$. As in Step 1 of the proof of [40, Proposition 4.2.4]), we deduce from the fact that $\sigma_i \geq (p+1)/(p-1)$ for $i \in \mathbb{B}_{\rm new}$ that Γ_w does not permute the new tails.

Suppose that there exists a tail $i \in \mathbb{B}$ such that Γ_w does not act trivially on $\bar{Y}_i$. Then [40, Lemme .2.6] implies that $\Gamma_w \cap G_i \neq \emptyset$. Since Γ_w is a p-group and p strictly divides the order of G, it follows that $\Gamma_w \cap G_i = \Gamma_w \cap G = P$.

Denote by $A^0_{G_0}(\bar{f}^{\rm aux})$ the set of tuples $(\gamma_0; \gamma_i \,|\, i \in \mathbb{B})$ satisfying the conditions (1) and (2) below. Here $\gamma_0 \in G_0$ and $\gamma_i : \bar{Y}_i \stackrel{\sim}{\to} \bar{Y}_i$ is an (outer) automorphism of the tail $\bar{Y}_i$ which commutes with the action of the decomposition group G_i and fixes η_i. (Recall that η_i is a fixed intersection point of $\bar{Y}_i$ with $\bar{Y}_0$.) The tuples $(\gamma_0; \gamma_i)$ are supposed to satisfy the following two conditions.

(1) The element $\gamma_0 \in G_0$ centralizes $H_0 \subset G_0$.

(2) The equality
$$\gamma_0^{-1} \circ \alpha \circ \gamma_0 = \gamma_i \circ \alpha \circ \gamma_i^{-1}$$
holds for all $\alpha \in G_i$.

This set is called the group of automorphisms of the special G_0-map $\bar{f}^{\rm aux}$ in [50, Section 2.2.4]. Lemma 2.17 of [50] states that we have an inclusion
$$\Gamma_w \;\longrightarrow\; A^0_{G_0}(\bar{f}^{\rm aux})/C_{G_0},$$
where C_{G_0} is the center of G_0.

Let $(\gamma_0; \gamma_i) \in A^0_{G_0}(\bar{f}^{\rm aux})$ be a nontrivial element whose class in $A^0_{G_0}(\bar{f}^{\rm aux})$ is contained in the image of $P \subset \Gamma_w$. Since $P \subset G_0$, we may take $\gamma_i = 1$ for $i \in \mathbb{B}$. Condition (2) implies that $P \subset C_{G_i}$ for all $i \in \mathbb{B}$. Since G_0 is generated by $(G_i)_{i \in \mathbb{B}}$, it follows therefore that $P \subset C_{G_0}$. This implies that $\sigma_i \in \mathbb{Z}$, for all $i \in \mathbb{B}$. Since the ramification of $f^{\rm aux}$ has prime-to-p order, it follows that $\sigma_i \neq 0$. But this contradicts the vanishing cycle formula (21). We conclude that Γ_w is trivial, and hence that $L^{\rm st} = \hat{K}_2(\lambda)$. □

Proposition 5.2.3 *The G-Galois cover $f : Y \to X$ may be defined over $\hat{K}_2(\lambda)$.*

Proof: This follows immediately from Lemma 5.2.2 and the construction of the auxiliary cover by formal patching ([40, Section 3.2]). Namely, it is shown in [40, Lemme 3.2.3] that there exists an étale cover $X' \to X$ covering $\bar{X}_0$ such that

$$(Y \times_X X' \longrightarrow X') = \mathrm{Ind}_{G_0}^G (Y^{\mathrm{aux}} \times_X X' \longrightarrow X').$$

The cover $f_R : Y_R \to X_R$ is obtained from $f_R^{\mathrm{aux}} : Y_R^{\mathrm{aux}} \to X$ by patching $\mathrm{Ind}_{G_0}^G Y^{\mathrm{aux}}$ together with suitable lifts of $\bar{f}_i|_{\bar{X}_i - \tau_i}$. The restriction of $\bar{f}_i$ to the complement of τ_i in $\bar{X}_i$ is tame for all $i \in \mathbb{B}$. Together with the fact that $\hat{K}_1(\lambda)$ is the maximally unramified extention of $\hat{K}(\lambda)$, this implies that $f : Y \to X$ may be defined over $\hat{K}_2(\lambda)$. □

5.3 The Swan conductor of a Hurwitz space We use the assumptions and notations of Section 5.2.

Let $\mathcal{H}_G/\mathbb{Q}_p$ be the inner Hurwitz space parameterizing G-Galois covers of $\mathbb{P}^1$ branched at four ordered points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$. There exists a smooth projective variety $\mathcal{H}_G^{\mathrm{adm}}$ such that the complement $\mathcal{H}_G^{\mathrm{adm}} - \mathcal{H}_G$ parameterizes admissible G-Galois covers ([48]). Let $\mathcal{H} = \mathcal{H}_f$ be the connected component of $\mathcal{H}_G^{\mathrm{adm}}$ such that the class of f corresponds to a point of $\mathcal{H}$. Then $\mathcal{H}$ is defined over a finite extention of $\mathbb{Q}_p$ which we denote by $\mathbb{Q}_p(\mathcal{H})$. We denote by $\pi : \mathcal{H} \to \mathbb{P}^1_\lambda$ the map which sends the class of a G-Galois cover to the branch point $x_3 = \lambda$. Let $\varpi : \mathbb{H} \to \mathbb{P}^1_\lambda$ be the Galois closure of π. We denote the Galois group of ϖ by Γ and the Galois group of $\mathbb{H}/\mathcal{H}$ by Γ_0. For what follows it is more convenient to extend the scalars to $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$. We write $K(\mathcal{H})$ for the function field of $\mathcal{H}$ over $\mathbb{C}_p$ and $K(\lambda)$ (resp. $K(\mathbb{H})$) for the corresponding function field of $\mathbb{P}^1_\lambda$ (resp. $\mathbb{H}$).

Let v_0 be the valuation of $K(\lambda)$ corresponding to $\bar{X}_0$. If v is a valuation of $K(\mathbb{H})$ above v_0, we write D_v (resp. I_v) for the decomposition group (resp. the inertia group) of v.

Theorem 5.3.1 *(a) For all valuations v of $K(\mathbb{H})$ above v_0, the index of $I_v \cap \Gamma_0$ in I_v is at most p.*

(b) There exists a v as in (a) such that the index of $I_v \cap \Gamma_0$ in I_v is p.

Proof: Proposition 5.2.3 implies that there exists an inclusion $K(\mathcal{H}) \to \hat{K}_2(D)$. Since $f : Y \to \mathbb{P}^1$ cannot be defined over $\hat{K}_1(\lambda)$, it follows that

$$\hat{K}_2(\lambda) = \hat{K}_1(\lambda) \cdot K(\mathcal{H}). \tag{63}$$

It follows that we may choose a valuation v of $K(\mathbb{H})$ above v_0 such that $\hat{K}_2(\lambda)$ is contained in the completion $K(\mathbb{H})_v$ of $K(\mathbb{H})$ with respect to v. Equation (63) implies that $K(\mathbb{H})_v = \hat{K}_1(\lambda) \cdot K(\mathbb{H})$. We conclude that

$$\mathrm{Gal}(K(\mathbb{H})_v, \hat{K}_2(\lambda)) = I_v \cap \Gamma_0.$$

This implies that the index of $I_v \cap \Gamma_0$ in I_v equals the degree of $\hat{K}_2(\lambda) \to \hat{K}_1(\lambda)$ which is p. This proves (a).

If v is a valuation of $K(\mathbb{H})$ above v_0 such that $\hat{K}_2(\lambda)$ is not contained in the completion $K(\mathbb{H})_v$, then clearly the inertia group I_v is contained in Γ_0. This proves (b). □

Theorem 5.3.2 *Let v be a valuation of $K(\mathbb{H})$ above v_0 such that the index of $I_v \cap \Gamma_0$ in I_v is p. There exists a nontrivial virtual character $\xi : I_v \to \tilde{\mathbb{Z}}$ with kernel $I_v \cap \Gamma_0$ such that the differential Swan conductor $\omega(\xi)$ equals the differential θ.*

Proof: Let v be as in the statement of the theorem, and let $\xi : I_v/I_v \cap \Gamma_0 \to \tilde{\mathbb{Z}}$ be a nontrivial character. We denote by $\omega(\xi)$ the corresponding differential Swan conductor. Recall from Section

5.1 that $\omega(\xi)$ is a differential form on the cover of $\mathbb{P}^1_\lambda$ corresponding to the extention of function fields $k(\lambda) \subset M_v$, where M_v is the residue field of $K(\mathbb{H})_v^{I_v}$. The group $D_v \cap \Gamma_0 / I_v \cap \Gamma_0$ leaves the differential form $\omega(\xi)$ invariant, therefore $\omega(\xi)$ descents to a differential form on the curve with function field

$$M_v^{D_v \cap \Gamma_0 / I_v \cap \Gamma_0}.$$

But this is just the field $k(C_0)$. Therefore $\omega(\xi)$ is the differential Swan conductor of the $\boldsymbol{\mu}_p$-torsor $k(D_0)/k(C_0)$. Example 5.1.5 implies that changing the irreducible character ξ multiplies $\omega(\xi)$ by a constant $b \in \mathbb{F}_p^\times$. Therefore for suitable choice of ξ we have that $\omega(\xi) = \theta$. □

6 The existence of covers with special reduction

The goal of this section to give sufficient conditions for the existence of covers with special reduction, satisfying Assumption 4.2.1.(b). We also give examples illustrating the results of Section 5. We mainly consider the case of $\mathrm{SL}_2(p)$ and $\mathrm{PSL}_2(p)$-Galois covers of the projective line branched at four points. The reason is that for these groups we known the reduction of three-point-covers ([13]; recalled in Section 6.1). This gives us good control over the possible signature of the reduction (Section 6.2).

The reason for considering $\mathrm{SL}_2(p)$-covers is the following. Let $f : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a $\mathrm{SL}_2(p)$-cover branched at there point defined over $\mathbb{C}$. A remarkable property of $\mathrm{SL}_2(p)$-covers of $\mathbb{P}^1$ branched at three points is that they are *rigid*. This means essentially that there is a unique cover if we fix the ramification, up to isomorphism. In [13] it is shown that this property implies that if $\tilde{f}$ has bad reduction, then $\sigma_i = (p+1)/(p-1)$ for all $i \in \mathbb{B}_{\mathrm{new}}$. In other words, in the terminology of Section 3.1 all new tail are nonsingular. A similar statement for the $\mathrm{PSL}_2(p)$-cover f can be easily deduced from this.

6.1 Reduction of three point covers In this section we recall some results of [13] on the reduction of $\mathrm{SL}_2(p)$-covers of the projective line branched at three points. We suppose that $p \geq 5$. Choose primitive $(p-1)$th root of unity $\zeta \in \mathbb{F}_p$ and a primitive $(p+1)$th root of unity $\tilde{\zeta} \in \mathbb{F}_{p^2}$. Define

$$\mathcal{C}(i) = \{\, A \in \mathrm{SL}_2(p) \,|\, \mathrm{tr}(A) = \zeta^i + \zeta^{-i} \,\}$$

and

$$\tilde{\mathcal{C}}(i) = \{\, A \in \mathrm{SL}_2(p) \,|\, \mathrm{tr}(A) = \tilde{\zeta}^i + \tilde{\zeta}^{-i} \,\}.$$

These are the conjugacy classes of $\mathrm{SL}_2(p)$ of nontrivial elements of order prime to p. We write pA and pB for the two conjugacy classes of order p. Suppose $\mathbf{C} = (C_0, C_1, \ldots C_r)$ is a tuple of conjugacy classes of $\mathrm{SL}_2(p)$ and $\mathbf{x} = (x_0, x_1, \ldots, x_r)$ is a tuple of pairwise distinct points of $\mathbb{P}^1_{\mathbb{C}}$. We write $\mathrm{Ni}_{r+1}(\mathbf{C}, \mathbf{x})$ for the set of isomorphism classes of $\mathrm{SL}_2(p)$-covers $Y \to \mathbb{P}^1_{\mathbb{C}}$ branched at $\mathbf{x}$ with class vector $\mathbf{C}$. This means that the canonical generator of inertia of some point of Y above x_i, with respect to a chosen compatible set of roots of unity, is an element of the conjugacy class C_i. More concretely,

$$\mathrm{Ni}_{r+1}(\mathbf{C}, \mathbf{x}) = \{(g_0, g_1, \ldots, g_r) \,|\, \mathrm{PSL}_2(p) = \langle g_i \rangle,\, g_i \in C_i,\, \prod_i g_i = 1\}/G.$$

Here G acts by uniform conjugation. We call two such covers $f_i : Y_i \to X$ isomorphic if there exists an $\mathrm{SL}_2(p)$-equivariant automorphism $\phi : Y_1 \to Y_2$ such that $f_1 = f_2 \circ \phi$. If $r + 1 = 3$, we suppose that $\mathbf{x} = \{\infty, 0, 1\}$ and omit $\mathbf{x}$ from the notation.

Now suppose that $r+1 = 3$ and let $\mathbf{C} = (C_0, C_1, C_2)$ be a triple of conjugacy classes of $\mathrm{SL}_2(p)$. Let $K/\mathbb{Q}_p$ be a finite extension such that the $\mathrm{SL}_2(p)$-covers parameterized by $\mathrm{Ni}_3(\mathbf{C})$ may be defined over K. Choose a prime $\wp$ of K above p, and replace K by its completion with respect to $\wp$. Define

$$\mathrm{Ni}_3^{\mathrm{bad}}(\mathbf{C}) = \{\, f \in \mathrm{Ni}_3(\mathbf{C}) \mid f \text{ has bad reduction } \}.$$

For $i = 0, 1, 2$, we define an integer a_i by $a_i = p - 1 - 2l$ if $C_i = \mathcal{C}(l)$, and $a_i = p + 1 - 2l$ if $C_i = \tilde{\mathcal{C}}(l)$, and $a_i = 0$ if $C_i \in \{pA, pB\}$. The following theorem is proved in [13].

Theorem 6.1.1 *(a) Suppose that $C_i \in \{pA, pB\}$, for some $i = 0, 1, 2$. Then $\mathrm{Ni}_3^{\mathrm{bad}}(\mathbf{C}) = \mathrm{Ni}_3(\mathbf{C})$ and*

$$|\,\mathrm{Ni}_3(\mathbf{C})| = \begin{cases} 1 & \text{if } a_0 + a_1 + a_2 < p - 1, \\ 2 & \text{if } a_0 + a_1 + a_2 = p - 1 \text{ and } C_i = \tilde{\mathcal{C}}(l) \text{ for some } i, \\ 0 & \text{otherwise} \end{cases}$$

(b) Suppose $C_i \notin \{pA, pB\}$ for all $i = 0, 1, 2$. Then $|\,\mathrm{Ni}_3(\mathbf{C})| \in \{0, 2\}$ and

$$|\,\mathrm{Ni}_3^{\mathrm{bad}}(\mathbf{C})| = \begin{cases} 2 & \text{if } a_1 + a_2 + a_3 < p - 1, \\ 2 & \text{if } a_1 + a_2 + a_3 = p - 1 \text{ and } C_i = \tilde{\mathcal{C}}(l) \text{ for some } i, \\ 0 & \text{otherwise} \end{cases}$$

(c) Suppose that $[f] \in \mathrm{Ni}_3^{\mathrm{bad}}(\mathbf{C})$. Then the deformation datum corresponding to the stable reduction $\bar{f}$ of f is special and multiplicative. It has signature $(a_0/(p-1), a_1/(p-1), a_2/(p-1))$ and all new tails are nonsingular.

Proof: It follows immediately from the definition that f has bad reduction if $C_i \in \{pA, pB\}$, for some i. The second part of (a) follows from rigidity ([13, Propostion 3.1.ii]) and the proof of [13, Theorem 5.6]. Part (b) follows from [13, Proposition 3.1.i] and [13, Theorem 5.6.b]. It is shown in [13, Corollary 5.4] that the new tails of a deformation datum corresponding to $[f] \in \mathrm{Ni}_3^{\mathrm{bad}}(\mathbf{C})$ are nonsingular. (Such deformation data are called *hypergeometric* in that paper.) The rest of (c) follows from the proof of [13, Theorem 5.6.b]. □

A consequence of the results of [13] is also a description of the $\mathrm{SL}_2(p)$-covers which may occur above the tails of the stable reduction of an $\mathrm{SL}_2(p)$-cover.

Definition 6.1.2 Let k be an algebraically closed field of characteristic $p > 0$. Let G be a finite group. A *G-tail cover* over k is a (not necessarily connected) cover $f_k : Y_k \to \mathbb{P}^1_k$ such that f_k is wildly branched at ∞ of order pn with n prime to p and tamely branched at no more than one other point. We say that f_k is a *primitive tail cover* if it is branched at two points. Otherwise, we call f_k a *new tail cover*.

The ramification invariant, σ, of a G-tail cover f_k is the ramification invariant of the unique wildly branched branch point.

Proposition 6.1.3 *Let $G = \mathrm{SL}_2(p)$, with $p > 3$.*

(a) Suppose that $f_k : Y_k \to \mathbb{P}^1_k$ is a connected, primitive G-tail cover with $0 < \sigma \leq 1$. Then the canonical generator of inertia of some point of Y_k above the tame branch point is contained in $\tilde{\mathcal{C}}(l)$, for some l, and $\sigma = (p + 1 - 2l)/(p - 1)$. These properties determine the tail cover uniquely, up to isomorphism.

(b) Suppose that $f_k : Y_k \to \mathbb{P}^1_k$ is a connected, new G-tail cover with $1 < \sigma \leq 2$. Then $\sigma = (p + 1)/(p - 1)$ and Y_k is the unique nonsingular projective curve given by the equation

$$xy^{p+1} - x^{p+1}y = 1. \qquad (64)$$

Proof: Part (a) is proved in [13, Proposition 5.5]. Part (b) is proved in [13, Proposition 5.3]. □

It is also easy to give equations for the primitive tail covers of Proposition 6.1.3.(a). A matrix $A \in \mathrm{SL}_2(p)$ acts on the curve defined by (64) by

$$\left(\begin{array}{cc} x & y \\ x^{p+1} & y^{p+1} \end{array} \right) \mapsto A \left(\begin{array}{cc} x & y \\ x^{p+1} & y^{p+1} \end{array} \right).$$

It is straightforward to deduce from Theorem 6.1.1 a corresponding result for $\mathrm{PSL}_2(p)$-covers of the projective line branched at three points, since one can lift every such $\mathrm{PSL}_2(p)$-cover to an $\mathrm{SL}_2(p)$-cover branched at three points.

6.2 A criterion for special reduction Suppose that $r+1=4$ and $p \geq 3$. Let G be a group whose order is strictly divisible by p. Let $f: Y \to \mathbb{P}^1_K$ be a G-Galois cover branched at $x_0=\infty, x_1=0, x_2=1, x_3=\lambda$ of order prime to p defined over a complete discrete valued field K of mixed characteristic p. We suppose that $(\mathbb{P}^1_K; x_i)$ is generic, i.e. λ is transcendental over $\mathbb{Q}_p$. The goal of this section is to prove a criterion for $G=\mathrm{SL}_2(p)$ which ensures that if f has bad reduction to characteristic p then f has special reduction.

Suppose that $f: Y \to \mathbb{P}^1_K$ has bad nonspecial reduction, and write $\bar{f}: \bar{Y} \to \bar{X}$ for its reduction, as usual. Choose an irreducible component $\bar{Y}_0$ of $\bar{Y}$ above the original component $\bar{X}_0$. Let $(\bar{g}_0, \omega)$ be the corresponding deformation datum.

As in the beginning of Section 3.4, we let $\mathbb{B}$ be the set of critical points of the deformation datum, and write $\sigma_i = \nu_i + a_i/(p-1)$ with $0 \leq a_i < p-1$. We denote by $\mathbb{B}_{\mathrm{prim}} = \{0,1,2,3\} \subset \mathbb{B}$ the set of primitive critical points and $\mathbb{B}_{\mathrm{new}} = \mathbb{B} - \mathbb{B}_{\mathrm{prim}}$ the set of new critical points. Since the reduction $\bar{f}: \bar{Y} \to \bar{X}$ is not special, it follows that either there is an $i \in \mathbb{B}_{\mathrm{prim}}$ such that $\nu_i \geq 1$ or there is an $i \in \mathbb{B}_{\mathrm{new}}$ such that $\nu_i \geq 2$. The vanishing cycle formula (Lemma 2.2.4.(a)) together with the assumption that $\mathbb{B}_{\mathrm{wild}} = \emptyset$ implies that

$$\sum_{i \in \mathbb{B}} a_i = p-1; \tag{65}$$

therefore there is a unique $i \in \mathbb{B}$ such that $\nu_i = 1$ if $i \in \mathbb{B}_{\mathrm{prim}}$ or $\nu_i = 2$ if $i \in \mathbb{B}_{\mathrm{new}}$. Let $\mathbb{B}_{\mathrm{ram}} = \{i \in \mathbb{B} \mid a_i \neq 0\}$.

Proposition 6.2.1 *Let f be as above. Suppose that $f: Y \to \mathbb{P}^1_K$ has bad nonspecial reduction. Then there exists an $i \in \mathbb{B}$ such that $\sigma_i \in \mathbb{Z}$.*

Proof: We use the notation from Section 3.4.

Lemma 3.4.2 states that the kernel, $\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0)^{\mathrm{loctriv}}$, of the local-global morphism

$$\mathrm{Def}\,(\bar{Y}_{\mathrm{sing}}, \mathcal{G}_0) \longrightarrow \prod_{i \in \mathbb{B}_{\mathrm{ram}}} \mathrm{Def}\,(\hat{Y}_i, \mathcal{G}_0)$$

has dimension zero. Therefore the local-global morphism is an isomorphism.

Let $R \in \mathfrak{C}_k$ be a local artinian k-algebra of equal characteristic p, and let $\mathcal{Y}_R$ be a $\mathcal{G}_0$-equivariant deformation of $\bar{Y}_{\mathrm{sing}}$. Denote by $\bar{Z}_{0,R} \to \bar{X}_{0,R}$ (resp. ω_R) the corresponding H_0-Galois cover (resp. logarithmic differential form). Let $j \in \mathbb{B}_{\mathrm{ram}}$, and let $z_j \in \bar{Z}_0$ be a point above $\tau_j \in \bar{X}_0$. We denote by $H_j \subset H_0$ the decomposition group of z_j and let $m_j = (p-1)/\gcd(p-1, a_j)$ be its order. There exists a local parameter t on $\bar{Z}_{0,R}$ at z_j such that $\mathcal{O}_{\bar{Z}_{0,R}, z_j} = R[[t]]$ and $h^*t = \xi(h) \cdot t$ for some character $\xi: H_j \to R^{\times}$. Following [51, Section 5.4], we say that $\bar{Y}_R$ is *j-special* if

$$\omega_R = t^{-1+(a_j+p-1)/\gcd(p-1,a_j)}(c_0 + c_1 t + \cdots)\,\mathrm{d}t,$$

where $c_i \in R$ and $c_0 \in R^{\times}$. In other words, the order at z_j of ω_R is equal to the order of ω.

We consider the subfunctor

$$\mathrm{Def}\,(\hat{Y}_j, \mathcal{G}_0)_{\mathrm{sp}} \subset \mathrm{Def}\,(\hat{Y}_j, \mathcal{G}_0)$$

of j-special local deformations. Lemma 5.13 of [51] implies that if $\sigma_j \notin \mathbb{Z}$ for all j, then a deformation is locally trivial if and only if it is j-special for all $j \in \mathbb{B}_{\mathrm{ram}}$. The idea is the following. It is clear that local triviality implies j-specialty for all $j \in \mathbb{B}_{\mathrm{ram}}$. Let $h_j = m_j\sigma_j$ the conductor of τ_j, and suppose that σ_j is not an integer. The $\boldsymbol{\mu}_p$-cover $\bar{Y}_{\mathrm{sing}} \to \bar{Z}_0$ may locally be given by a Kummer equation $y^p = v$, where

$$v = 1 + x^{h_j} + \text{higher order terms}.$$

Let $n' = \gcd(p-1, m_j)$ and $n = m_j/n'$. Since σ_j is not an integer it follows that $n \neq 1$. Let $\sigma \in H_0$ be an automorphism of order n. We may choose the parameter x such that $\sigma(x) = \zeta_n x$ for some primitive mth root of unity $\zeta_m \in \mathbb{F}_p$. Let $\bar{Y}_R$ be an $\mathcal{G}_0$-equivariant deformation of $\bar{Y}_{\mathrm{sing}}$ which is j-special, but not locally trivial around z_j. Then the $\boldsymbol{\mu}_p$-torsor $\bar{Y}_R \to \bar{Z}_{0,R}$ is given by an equation $y^p = v_R$, where

$$v_R = c + x^{h_j} + \cdots,$$

for some $c \in R^\times$, since the deformation is j-special. The fact that σ does not commute with the $\boldsymbol{\mu}_p$-action, implies that

$$v_R^\sigma = v_R^{\zeta_m} w^p.$$

But this implies that c is a pth-power in k which contradicts the fact that the deformation not locally trivial. Note however, that the deformation becomes locally trivial after pull back via a purely inseparable extension. □

In the rest of this section we suppose that $G = \mathrm{SL}_2(p)$. A similar consideration holds for $\mathrm{PSL}_2(p)$, and probably for other linear groups as well. Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover over K branched at $x = \infty, 0, 1, \lambda$ with class vector $\mathbf{C} = (C_0, C_1, C_2, C_3)$, where we suppose that $C_i \neq pA, pB$ for all i. As in Section 6.1, for $i = 0, 1, 2, 3$ we define

$$a_i = \begin{cases} p - 1 - 2l & \text{if } C_i = \mathcal{C}(l), \\ p + 1 - 2l & \text{if } C_i = \tilde{\mathcal{C}}(l). \end{cases} \qquad (66)$$

The following proposition is in some sense an analog of Theorem 6.1.1 for G-Galois cover of $\mathbb{P}^1_K$ branched at four points. Since four-point covers are not rigid, the statement is not as strong as for three-point covers. The proposition gives a criterion on the class vector $\mathbf{C}$ which guarantees that all G-covers with bad reduction have special reduction.

Proposition 6.2.2 *Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover with class vector $\mathbf{C} = (C_0, C_1, C_2, C_3)$. Suppose that $f : Y \to \mathbb{P}^1_K$ has nonspecial bad reduction to characteristic p.*

(a) We have

$$a_0 + a_1 + a_2 + a_3 \leq p - 1.$$

If $a_0 + a_1 + a_2 + a_3 = p - 1$, there exists an $i \in \mathbb{B}_{\mathrm{prim}} = \{0, 1, 2, 3\}$ such that $C_i = \tilde{\mathcal{C}}(l)$.

(b) Moreover, for all $i \in \mathbb{B}_{\mathrm{prim}}$ we have $\sigma_i = a_i/(p-1)$. For all $i \in \mathbb{B}_{\mathrm{new}}$ except possibly one, we have $\sigma_i = (p-1)/(p-1)$.

Proof: Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover with class vector $\mathbf{C} = (C_0, C_1, C_2, C_3)$. Suppose that $f : Y \to \mathbb{P}^1_K$ has nonspecial bad reduction to characteristic p, and denote the stable reduction by $\bar{f} : \bar{Y} \to \bar{X}$. Proposition 6.2.1 implies that one of the following two cases occurs.

- There exists a unique $i \in \mathbb{B}_{\mathrm{prim}} = \{0, 1, 2, 3\}$ such that $\sigma_i = 1$.

- There exists a unique $i \in \mathbb{B}_{\rm new}$ such that $\sigma_i = 2$.

Let $i \in \mathbb{B}_{\rm new}$ (resp. $i \in \mathbb{B}_{\rm prim}$). By the above, it follows that $\sigma_i \leq 2$ (resp. $\sigma_i \leq 1$). Therefore there is a unique tail $\bar{X}_i$ of $\bar{X}$ which intersects $\bar{X}_0$ in the critical point τ_i.

If $i \in \mathbb{B}_{\rm new}$, the decomposition group $G_i \subset G$ of an irreducible component $\bar{Y}_i$ of $\bar{Y}$ above $\bar{X}_i$ is a quasi-p group, i.e. a group which is generated by its Sylow p-subgroup. Therefore G_i is either the full group $\mathrm{SL}_2(p)$ or a cyclic group of order p. Proposition 6.1.3 implies therefore that either $\sigma_i = 2$ and $G_i \simeq \mathbb{Z}/p$ or $\sigma_i = (p+1)/(p-1)$ and $G_i = \mathrm{SL}_2(p)$.

If $i \in \mathbb{B}_{\rm prim}$, Proposition 6.1.3 implies that $\sigma_i = a_i/(p-1)$, where a_i is as defined in (66). Moreover, also in this case the tail cover is uniquely determined up to isomorphism by the conjugacy class C_i. This proves (b). Part (a) now follows from the vanishing cycle formula (65). If $a_0 + a_1 + a_2 + a_3 = p - 1$, there are no new critical points. The condition on the conjugacy classes follows from the observation that $\bar{Y}$ should be connected. □

For completeness, we state the following analog of Proposition 6.2.2 for covers with special bad reduction.

Lemma 6.2.3 *Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover branched at $x = \infty, 0, 1, \lambda$ with special bad reduction. Then $\sigma_i = a_i/(p-1)$ for all $i \in \mathbb{B}_{\rm prim}$ and $\sigma_i = (p+1)/(p-1)$ for all $i \in \mathbb{B}_{\rm new}$.*

Proof: This follows immediately from Proposition 6.1.3, since $0 < \sigma_i < 1$ (resp. $1 < \sigma_i < 2$) for all $i \in \mathbb{B}_{\rm prim}$ (resp. $i \in \mathbb{B}_{\rm new}$), by definition of special reduction. □

Lemma 6.2.3 and Proposition 6.2.2.(b) together with Proposition 2.4.1 imply the following proposition.

Proposition 6.2.4 *Let $(\bar{g}_0, \omega)$ be a deformation datum with:*

- $r + 1 = |\mathbb{B}_{\rm prim}| = 4$,
- $0 \leq \sigma_i = a_i/(p-1) \leq 1$ *with a_i even for* $i \in \mathbb{B}_{\rm prim} = \{0, 1, 2, 3\}$,
- $\sigma_i \in \{(p+1)/(p-1), 2\}$ *for* $i \in \mathbb{B}_{\rm new}$.

Then there exists an $\mathrm{SL}_2(p)$-cover $f : Y \to \mathbb{P}^1_K$ branched at four points which has bad reduction which gives rise to the deformation datum $(\bar{g}_0, \omega)$.

Proof: This follows from a standard formal-patching argument, as in the proof of Corollary 3.4.5. □

6.3 The p-cusps In this section we give a sufficient condition for Assumption 4.2.1.(b) to be satisfied. Recall that this condition states that the dual Hasse invariant Φ is nonzero.

Let G be a finite group whose order is strictly divisible by p. Let $\mathcal{H}/\mathbb{Q}_p(\mathcal{H})$ be a connected component of the inner Hurwitz space parameterizing G-Galois covers of $\mathbb{P}^1$ branched at four points, as in Section 5.3. Write $\varpi : \mathcal{H} \to \mathbb{P}^1_\lambda$ for the natural map. We call the points $\varpi^{-1}(\{0, 1, \infty\})$ the *cusps* of $\mathcal{H}$. Recall that they parameterize admissible G-covers.

Let K be a complete discrete valuation field of characteristic zero whose residue field, k, is an algebraically closed field of characteristic $p > 0$. Let $(X^{\rm adm}; x_0, x_1, x_2, x_3)$ be a stably marked curve of genus zero. It consist of two irreducible components X', X'' which meet in a unique point μ. Let $f^{\rm adm} : Y^{\rm adm} \to X^{\rm adm}$ be an admissible G-Galois cover branched at x_0, x_1, x_2, x_3. Choose a point $\rho \in Y^{\rm adm}$ above μ, and write Y', Y'' for the irreducible components of $Y^{\rm adm}$ which pass through ρ, where we suppose that Y' (resp. Y'') maps to X' (resp. X''). Write $f' : Y' \to X'$ and $f'' : Y'' \to X''$ for the restriction. Then f' and f'' are branched at at most three points.

Definition 6.3.1 We say that $f_K^{\rm adm}$ is a *p-cusps* of the Hurwitz space $\mathcal{H}$ if the ramification index of μ in $f_K^{\rm adm}$ is equal to p.

Proposition 6.3.2 *Let $f: Y \to \mathbb{P}^1_K$ be a G-Galois cover branched at four points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$ of order prime to p which has bad reduction. Suppose that f specializes in equal characteristic zero to a p-cusp.*

(a) The cover f has special reduction to characteristic p.

(b) Assumption 4.2.1.(b) is satisfied.

Proof: Let $f: Y \to \mathbb{P}^1_K$ be as in the statement of the proposition and write $\bar{f}: \bar{Y} \to \bar{X}$ for its stable reduction. Our assumptions imply in particular that $(\mathbb{P}^1_K; x_i)$ is generic, hence $(\mathbb{P}^1_K; x_i)$ has good reduction. Choose a component $\bar{Y}_0$ of $\bar{Y}$ above the original component $\bar{X}_0$ and let $(\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0, \omega)$ be the corresponding deformation datum. We write $\mathbb{B}$ for the set of critical points of $(\bar{g}_0, \omega)$.

Let $f^{\rm adm} : Y^{\rm adm} \to X^{\rm adm}$ be the p-cusp to which f specializes, and write $f' : Y' \to X'$ and $f'' : Y'' \to X''$ for the associated three point covers and let G' (resp. G'') be the decomposition group of Y' (resp. Y'').

Since $f^{\rm adm}$ is a p-cusp, the covers f' and f'' both have bad reduction. Denote their stable reduction by $\bar{f}' : \bar{Y}' \to \bar{X}'$ and $\bar{f}'' : \bar{Y}'' \to \bar{X}''$. Let $(\bar{g}_0' : \bar{Z}_0' \to \bar{X}_0', \omega')$ (resp. $(\bar{g}_0'' : \bar{Z}_0'' \to \bar{X}_0'', \omega'')$) be the deformation datum of $\bar{f}'$ (resp. $\bar{f}''$). As usual, we write $\mathbb{B}'$ and $\mathbb{B}''$ for the set of critical points of $(\bar{g}_0', \omega')$ and $(\bar{g}_0'', \omega'')$. Since the ramification index of μ is p, we have that $\sigma_{\mu'} = \sigma_{\mu''} = 0$. Moreover, the point μ specializes to a point μ' (resp. μ'') on the original component $\bar{X}_0'$ (resp. $\bar{X}_0''$). This means that we may define a G-equivariant map of semistable curve $\bar{f}^{\rm adm} : \bar{Y}^{\rm adm} \to \bar{X}^{\rm adm}$ by suitably identifying points in the fiber above μ' in $\mathrm{Ind}_{G'}^G \bar{Y}'$ with points in the fiber above μ'' in $\mathrm{Ind}_{G''}^G \bar{Y}''$. Comparing the genus of $\bar{Y}^{\rm adm}$ and $Y^{\rm adm}$ as in the proof of [12, Proposition 2.5.3.(b)], we find that $g(Y^{\rm adm}) = g(\bar{Y}^{\rm adm})$, hence $\bar{f}^{\rm adm}$ is the reduction of $f^{\rm adm}$ (compare to [12, Section 2.5]).

Since f specializes to $f^{\rm adm}$, it follows that also $\bar{f}$ specializes to $\bar{f}^{\rm adm}$. The proposition will now follow by comparing the genus of $\bar{Y}^{\rm adm}$ with the genus of $\bar{Y}$. The vanishing formula (Lemma 2.2.4) together with the assumption that the ramification indices of f are prime to p implies that

$$\sum_{i \in \mathbb{B}'} a_i = p - 1, \qquad \sum_{i \in \mathbb{B}''} a_i = p - 1. \tag{67}$$

Since $g(Y^{\rm adm}) = g(\bar{Y}^{\rm adm}) = g(Y)$ it follows that no new critical point of $(\bar{g}_0, \omega)$ specialize to the point μ in $\bar{X}^{\rm adm}$. Therefore a new critical point τ_i of $(\bar{g}_0, \omega)$ specializes either to a new critical point τ_i for $i \in \mathbb{B}'_{\rm new} \cup \mathbb{B}''_{\rm new}$ on $\bar{X}^{\rm adm}$ or to one of the points $\tau_0, \tau_1, \tau_2, \tau_3$ on $\bar{X}_0' \coprod \bar{X}_0'' - \{\mu', \mu''\}$. Since $\sigma_{\mu'} = \sigma_{\mu''} = 0$, it follows from (67) that

$$\sum_{i \in \mathbb{B}} a_i = \sum_{i \in \mathbb{B}'} a_i + \sum_{i \in \mathbb{B}''} a_i = 2(p-1).$$

This implies that f has special reduction.

To prove (b), we consider the cover $\bar{g}_0^{\rm adm} : \bar{Z}_0^{\rm adm} \to \bar{X}_0^{\rm adm}$. It is an admissible cover which is the specialization of $\bar{g}_0 : \bar{Z}_0 \to \bar{X}_0$. Its restriction to $\bar{X}_0'$ (resp. $\bar{X}_0''$) is induced from $\bar{g}_0'$ (resp. $\bar{g}_0''$). Since the ramification index of μ is p, it follows that the reduction of $\mu \in \bar{X}_0^{\rm adm}$ is unramified in $\bar{g}_0^{\rm adm}$. Equation (67) implies that $\dim_k H^1(\bar{Z}_0', \mathcal{O})_\chi = \dim_k H^1(\bar{Z}_0'', \mathcal{O})_\chi = 0$. Therefore it trivially follows that the Frobenius morphism F is an isomorphism on $H^1(\bar{Z}_0', \mathcal{O})_\chi$ and $H^1(\bar{Z}_0'', \mathcal{O})_\chi = 0$. It is well known that this implies that $F : H^1(\bar{Z}_0^{\rm adm}, OO)_\chi \to H^1(\bar{Z}_0^{\rm adm}, OO)_\chi$ is an isomorphism as well ([6, Lemma 1.3]). This implies that $F : H^1(\bar{Z}_0, \mathcal{O})_\chi \to H^1(\bar{Z}_0, \mathcal{O})_\chi$ is an isomorphism. Part (b) follows. □

Corollary 6.3.3 *Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover with bad reduction which specializes to a p-cusp $f^{\rm adm} : Y^{\rm adm} \to X^{\rm adm}$. Let $\pi : B_0 \to \mathbb{P}^1_\lambda$ be the accessary-parameter cover defined by the deformation datum of the stable reduction of f. Then $[f^{\rm adm}]$ correspond to a point $b_0 \in \pi^{-1}(\{0,1,\infty\}) \subset \Sigma_0$ with logarithmic monodromy.*

Proof: Let f and $f^{\rm adm}$ be as in the statement of the corollary. Proposition 6.3.2 implies that f has special reduction. Let $(\mathcal{E},\nabla)$ be the flat vector bundle corresponding to f. The construction of the cover $\bar{f}^{\rm adm}$ in the proof of Proposition 6.3.2 defines a point $b_0 \in \pi^{-1}(\{0,1,\infty\} \subset \Sigma_0$. Moreover, the proof implies that $\mathrm{ord}_{b_0}(\Phi) \equiv \mathrm{ord}_{b_0}(\Phi_*) \equiv 0 \bmod p$. Let $\alpha_{b_0}, \beta_{b_0}$ be the local exponents of the differential equation corresponding to $(\mathcal{E},\nabla)$. Proposition 4.5.2 implies that $\alpha_{b_0} = \beta_{b_0} = 0$. Therefore b_0 has logarithmic monodromy (this notion was defined in Section 4.1). □

6.4 An example In this section we consider a concrete example of $\mathrm{SL}_2(p)$-covers with bad reduction, and discuss what can be said about the reduction of the corresponding Hurwitz spaces.

Let $p \geq 7$ be a prime number and $\mathbf{a} = (a_0,a_1,a_2,a_3) := (p-5,p-5,2,2)$. It follows that $d = 2(p-1) - (a_0+a_1+a_2+a_3) = 2$.

We start by computing the possible deformation data with signature $\boldsymbol{\sigma} = (\sigma_i = a_i/(p-1))$. We want to find a solution u of degree 2 of the differential equation

$$L(u) = P_0u'' + P_1u' + P_2u = 0, \quad \text{with} \tag{68}$$

$$P_0 = x(x-1)(x-\lambda), \quad P_1 = 2x^2 + x(\lambda+1) - 4\lambda, \quad P_2 = -6x + \beta, \tag{69}$$

as in Section 3.3. One checks that the accessary parameter β should satisfy

$$(\beta + 8\lambda)(14\beta\lambda + 56\lambda + \beta^2 + \beta) = 0.$$

If $p = 7$ we find that β equals either 0, -1 or $-\lambda$. The corresponding deformation datum is in all three cases essentially the same. If $p \geq 11$ the polynomial $14\beta\lambda + 56\lambda + \beta^2 + \beta$ is irreducible in $\mathbb{F}_p[\beta,\lambda]$, and there are two really different possibilities for the accessary parameter.

From now on we suppose that $p \geq 11$, since this is the more interesting case. Let $\pi_2 : B_0^2 \to \mathbb{P}^1_\lambda$ be the cover defined by

$$\lambda = -\frac{\beta(\beta+1)}{14(\beta+4)}. \tag{70}$$

Note that it is ramified in the points with $\beta^2 + 8\beta + 4 = 0$, and the genus of B_0^2 is zero. Then

$$u(x) = 7(\beta+4)^2x^2 - 7\beta(\beta+4)x + 2\beta(\beta+1)$$

is a solution of the differential equation (68). It is the unique such solution, up to multiplication with an element of $k(B_0)$.

Let $\bar{g}_0 : \bar{Z}_0 \to \mathbb{P}^1_k$ be the $(p-1)/2$-cyclic cover of smooth projective curves corresponding to u and $\mathbf{a}$, i.e. $\bar{Z}_0$ is the normalization of a connected component of the curve given by the Kummer equation

$$z^{p-1} = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3}u^2.$$

The curve $\bar{Z}_0$ lives over $B_0^2 - \Sigma_0$, where $\Sigma_0 = \pi^{-1}(\{0,1,\infty\}) = \{0,-1,-7,-8,-4,\infty\}$. Write $\omega_0 = z\,\mathrm{d}x/x(x-1)(x-\lambda) \in H^0(\bar{Z}_0,\Omega)$.

Up to multiplying by an element of $\mathbb{F}_p^\times$, the Hasse invariant and the dual Hasse invariant are given by

$$\Phi_* = \frac{\psi}{(\beta+4)^6}, \quad \Phi = \frac{\beta(\beta+8)(\beta^2+8\beta+4)\psi}{(\beta+4)^2},$$

where

$$\psi = \beta^4 + 16\beta^3 + 141\beta^2 + 616\beta + 3136$$

is the polynomial whose zeros are the supersingular points. It follows that assumptions (a)–(e) of Section 5.2 are satisfied for the deformation datum corresponding to $(\bar{g}_0, \omega_0)$. Note that $\pi^2 : B_0^2 \to \mathbb{P}^1_\lambda$ is unramified at the supersingular points, therefore the Kodaira–Spencer map is everywhere nonzero (Theorem 4.7.3). Let $(\mathcal{E}, \nabla)$ be the corresponding pseudo elliptic bundle. It follows from Lemma 3.2.1 and Proposition 4.2.2 that Σ_0 is the set of singularities of the differential equation corresponding to the pseudo elliptic bundle $(\mathcal{E}^2, \nabla^2)$.

Let $D = \partial/\partial\beta$. One computes that the differential equation satisfied by ω_0 is given by

$$\nabla(D)^2\omega_0 + \delta_1^*\nabla(D)\omega_0 + \delta_0^*\omega_0 = 0 \in H^1_{\mathrm{dR}}(\bar{Z}_0, \Omega)_\chi,$$

where

$$\begin{aligned} \delta_1^* &= \frac{4}{\beta+4} + \frac{2}{\beta} + \frac{1}{\beta+7} + \frac{1}{\beta+1} + \frac{2}{\beta+8}, \\ \delta_0^* &= \frac{73}{28\beta} - \frac{13}{7(\beta+7)} + \frac{13}{7(\beta+1)} - \frac{18}{(\beta+4)^2} - \frac{73}{28(\beta+8)}. \end{aligned}$$

It follows that the local exponents at $\beta = 0, -1, -7, -8, -4, \infty$ are $0, -1; 0, 0; 0, 0; 0, -1; -6, 3; 2, 7$.

Let

$$v = \frac{\partial\lambda/\partial\beta}{\Phi_*\Phi\lambda(\lambda-1)} = -\frac{14(\beta+4)^8}{\beta^2(\beta+8)^2(\beta+1)(\beta+7)\psi^2}.$$

One computes that $-\mathrm{Res}_0(v) = -\mathrm{Res}_{-1}v = \mathrm{Res}_{-7}v = \mathrm{Res}_{-8}v = 3/67228$, and that the residue of v at all other points of B_0^2 is zero. It follows that

$$\begin{aligned} D^{p-1}v &= \frac{3}{67228}\left(\frac{1}{\beta^p} + \frac{1}{(\beta+1)^p} - \frac{1}{(\beta+7)^p} - \frac{1}{(\beta+8)^2}\right) = \frac{3(\beta^2+8\beta+4)^p}{4802\beta^p(\beta+1)^p(\beta+8)^p(\beta+7)^p} \\ &= \frac{-3}{67228}\frac{(\partial\lambda/\partial\beta)^p}{\lambda^p(\lambda-1)^p} =: -W^p. \end{aligned}$$

The corresponding deformation datum is given by

$$y^{p-1} = \frac{W^p}{v} = -\frac{3}{343}\frac{(\beta^2+8\beta+4)^p\psi^2}{\beta^{p-2}(\beta+1)^{p-1}(\beta+7)^{p-1}(\beta+8)^{p-2}(\beta+4)^8}, \qquad \theta = y\,\mathrm{d}\beta.$$

The signature of the deformation datum is

b	0	-1	-7	-8	-4	∞	$\beta^2+8\beta+4=0$
σ_b	$\frac{1}{p-1}$	0	0	$\frac{1}{p-1}$	$\frac{p-9}{p-1}$	$\frac{p-5}{p-1}$	$\frac{2p-1}{p-1}$.

We now compute the other possibility for a special deformation datum of signature $\boldsymbol{\sigma}$. Let $B_0^1 = \mathbb{P}^1_\lambda$ be the curve corresponding to $\beta = -6\lambda$. The corresponding solution of (68) is $u = x^2 - 2\lambda x + \lambda$. We leave it to the reader to compute that this defines the following deformation datum

$$y^{p-1} = -\frac{(5\lambda^2-5\lambda+1)^2}{\lambda^{p-2}(\lambda-1)^{p-2}}, \qquad \theta = y\,\mathrm{d}\lambda.$$

Note that the singularities are $\Sigma_0 = \{\lambda = 0, 1, \infty\}$ and the supersingular points are $\Sigma_1 = \{5\lambda^2 - 5\lambda + 1 = 0\}$. The signature of the deformation datum is

b	0	1	∞
σ_b	$\frac{1}{p-1}$	$\frac{1}{p-1}$	$\frac{p-3}{p-1}$.

Let $G = \mathrm{SL}_2(p)$. Choose a primitive $(p-1)$th root of unity $\zeta \in \mathbb{F}_p^\times$, and let $C_1 = \mathcal{C}((p-3)/2)$ and $C_2 = \mathcal{C}(2)$ be the conjugacy classes of $\mathrm{SL}_2(p)$ defined in Section 6.1. Proposition 6.2.4 implies that there exists a G-Galois cover $f : Y \to \mathbb{P}^1_K$ with class vector $\mathbf{C} = (C_1, C_1, C_2, C_2)$ branched at four points x_0, x_1, x_2, x_3 which has bad reduction and whose reduction gives rise to the deformation datum $(\bar{g}_0, \omega)$ defined above. (To show that one may choose the class vector as stipulated, one uses Proposition 6.1.3.)

Suppose that $f : Y \to \mathbb{P}^1$ is any G-Galois cover with class vector $\mathbf{C}$ and bad reduction. It follows from Proposition 6.2.2 that f has special reduction. Let $(\bar{g}_0, \omega)$ be the special deformation datum of its reduction and write (σ_i) for the signature. Lemma 6.2.3 implies that $\sigma_i = a_i/(p-1)$ for $i \in \mathbb{B}_{\mathrm{prim}} = \{0, 1, 2, 3\}$, where a_i are as defined in the beginning of this section. Moreover $\sigma_i = (p+1)/(p-1)$ for all $i \in \mathbb{B}_{\mathrm{new}}$. This implies that all G-Galois cover with class vector $\mathbf{C}$ have special reduction. Hence we described all possible deformation data, in our situation.

We now describe the cusps with bad reduction, using the notation introduced in Section 6.3. Let $\mathcal{H}(\mathbf{C})/\mathbb{Q}_p(\mathbf{C})$ be the Hurwitz space parameterizing G-Galois covers of $\mathbb{P}^1$ branched at four points $x_0 = \infty, x_1 = 0, x_2 = 1, x_3 = \lambda$ with class vector $\mathbf{C}$. Let $f^{\mathrm{adm}} : Y^{\mathrm{adm}} \to X^{\mathrm{adm}}$ be an admissible G-Galois cover corresponding to a cusp of $\mathcal{H}(\mathbf{C})$. As in Section 6.3, we write $f' : Y' \to X'$ (resp. $f'' : Y'' \to X''$) for the corresponding three-point covers. Suppose that at least one of f' and f'' has bad reduction. Let us consider the cusps above $\lambda = 0$ and suppose that x_1, x_3 specialize to X' and x_0, x_2 specialize to X''. Write μ for the point of X^{adm} where X' and X'' intersect, and $\mathbf{C}' = (C_1, C_2, C_3)$ for the class vector of both f' and f''. Here C_3 is the conjugacy class corresponding to the ramification of μ. We use that f^{adm} is admissible and $g \sim g^{-1}$ in G. Theorem 6.1.1 implies the following.

Lemma 6.4.1 *Let $f' : Y' \to X'$ be a (possibly disconnected) G-Galois cover with class vector (C_1, C_2, C_3). Then f' has bad reduction if and only if $C_3 \in \{pA, pB, \mathcal{C}((p-3)/2), \tilde{\mathcal{C}}((p-1)/2)\}$ and p divides the order of the decomposition group of a connected component of Y'.*

This gives a concrete way of computing the number of $\mathrm{SL}_2(p)$-covers with bad reduction, similar to the result of [12, Section 5].

Let $\mathcal{H}$ be a connected component of $\mathcal{H}(\mathbf{C})$ and suppose that $g(\mathcal{H}) > 1$. Write $\bar{\mathcal{H}}$ for the stable reduction of $\mathcal{H}$. (Contrary to what we did so far, we do not consider a marking on $\mathcal{H}$.) The cover $\varpi : \mathcal{H} \to \mathbb{P}^1_\lambda$ extends to a map $\bar{\varpi} : \bar{\mathcal{H}} \to \mathbb{P}^1_{\lambda,k}$ which will not be finite in general. The irreducible components of $\mathcal{H}$ which map subjectively to $\mathbb{P}^1_{\lambda,k}$ are called the *horizontal components*. The irreducible components of $\bar{\mathcal{H}}$ which are mapped to a point on $\mathbb{P}^1_{\lambda,k}$ are called the *vertical components*. Let $f : Y \to \mathbb{P}^1_K$ be a G-Galois cover corresponding to a point of $\mathcal{H}$ above the generic point of $\mathbb{P}^1_\lambda$. Suppose that f has bad reduction, and let $\bar{\mathcal{H}}(f)$ be the corresponding horizontal component of the reduction. We call such component a *bad horizontal component*. The *bad degree* is the the total degree of all bad components over $\mathbb{P}^1_{\lambda,k}$. The deformation datum of f defines an accessary-parameter cover $\pi = \pi(f) : B_0(f) \to \mathbb{P}^1_{\lambda,k}$. Analogous to [12, Theorem 3.1.2], it may be shown that we obtain an isomorphism between $B_0(f)$ and the underlying reduced subscheme of $\bar{\mathcal{H}}(f)$. This relies on the deformation theory of $\boldsymbol{\mu}_p$-torsors, as explained in Section 5.2 together with the arguments of Section 5.3. Therefore one may count the number of bad components, by using the description of the cusps with bad reduction we gave above. It would be interesting to see how much information this gives on the reduction on the Galois closure of the Hurwitz space, as in [8].

To make the previous discussion more concrete, suppose that $p = 11$. For convenience, we divide out by the center of G, i.e. we suppose that $G = \mathrm{PSL}_2(p)$. Let $\mathbf{C} = (C_1, C_1, C_2, C_2)$ be as above, and let $\mathcal{H}(\mathbf{C})$ be the Hurwitz space parameterizing G-Galois covers with class vector $\mathbf{C}$. Using the computer program GAP, one computes that $\mathcal{H}(\mathbf{C})$ has three connected component which we denote by H_1, H_2 and H_3. The degree of $\varpi_i : H_i \to \mathbb{P}^1_\lambda$ is $164, 110, 328$, for $i = 1, 2, 3$. The Galois group Γ_i of the Galois closure of ϖ_i is isomorphic to S_{82}, A_{55}, A_{82}.

Write

$$\mathrm{Ni}(\mathbf{C}) = \{(g_0, g_1, g_2, g_3) \,|\, g_i \in C_i \ \text{ and } G =< g_i >\}/G$$

for the set of Nielsen classes. Here G acts of the tuples (g_0, g_1, g_2, g_3) by uniform conjugacy. By Riemann existence theorem, the Nielsen classes correspond to the G-Galois covers over a fixed marked curve $(\mathbb{P}^1_K; x_i)$. In our special case, we write $\mathrm{Ni}(\mathbf{C})_i$ for the subsets of $\mathrm{Ni}(\mathbf{C})$ corresponding to the connected component $\mathcal{H}_i$ of $\mathcal{H}$. It is well known that these are the orbits under the pure Artin braid group $\mathcal{B}^{(4)}$ [47]. It is well known how to describe the cusps in terms of the action of the braid group: the cusps above $\lambda = 0$ (resp. $\lambda = 1$, resp. $\lambda = \infty$) correspond to the orbits of $\mathrm{Ni}(\mathbf{C})$ under certain concrete elements b_0 (resp. b_1, resp. b_∞) of the pure Artin braid group, see for example [48].

As an example, we consider the cover $\varpi_3 : \mathcal{H}_3 \to \mathbb{P}^1_\lambda$. The following table gives a list of the cusps above $\lambda = 0 \in \mathbb{P}^1_\lambda$. Here $|G'|$ (resp. $|G''|$) is the order of the decomposition of a connected component of Y' (resp. Y'') in the notation we explained above, n is the ramification index of the singular point μ, and 'number' is the number of such cusps and 'ram' its ramification index in ϖ_3. The last entry labels the different types of cusps.

$\lvert G'\rvert$	$\lvert G''\rvert$	n	number	ram	label
5	660	5	4	1	$5A$
660	5	5	4	1	$5B$
55	660	5	16	5	$5C$
660	55	5	16	5	$5D$
60	60	2	10	2	$2A$
60	660	3	8	3	$3A$
660	60	3	8	3	$3B$
660	660	6	8	6	$6A$
660	660	11	4	11	$11A$

The cusps labeled $11A$ are p-cusps (Section 6.3). They correspond to admissible covers $f^{\mathrm{adm}} : Y^{\mathrm{adm}} \to X^{\mathrm{adm}}$ in characteristic zero; the restriction of f^{adm} to both X' and X'' is a cover with class vector (C_1, C_2, C_3), where $C_3 \in \{pA, pB\}$. In particular, both f' and f'' has bad reduction and the cusps $[f^{\mathrm{adm}}]$ specializes to to a bad horizontal component. Corollary 6.3.3 implies that the reduction of the p-cusp $[f^{\mathrm{adm}}]$ corresponds to a logarithmic singularity on a horizontal bad component. We have seen that the underlying reduced subspace of a horizontal bad component is isomorphic to either B_0^1 or B_0^2. Since the pseudo elliptic bundle $(\mathcal{E}_1, \nabla_1)$ corresponding to B_0^1 does not have any logarithmic singularities, it follows that a p-cusp $[f^{\mathrm{adm}}]$ specializes to a horizontal bad component whose underlying reduced subscheme is isomorphic to B_0^2. In particular, it follows that the number N_2 of such horizontal bad components, counted with multiplicity, is 4 which is the number of p-cusps. To compute the multiplicity, one needs a more precise analyses of the universal deformation rings (cf. [12, Section 3]).

To compute the number N_1 of horizontal bad components whose underlying reduced subscheme is isomorphic to B_0^1, we need to consider the other cusps with bad reduction. Lemma 6.4.1 implies that the cusps with labels $2A, 3A, 3B$ have admissible reduction. It remains to consider the cusps with label $5A, 5B, 5C, 5D, 6A$. A cusp $[f^{\mathrm{adm}}]$ is called a *bad cusp* if either f' or f'' have bad reduction to characteristic p.

Lemma 6.4.2 *(a) All cusps of label $5C$ and $5D$ are bad cusps.*

(b) *Half of the cusps of label* $5A$ *and* $5B$ *are bad cusps.*

(c) *A quarter of the cusps of label* $6A$ *are bad cusps.*

Proof: Since $\mathcal{H}_3$ is the only connected component of $\mathcal{H}(\mathbf{C}$ whose degree over $\mathbb{P}^1_\lambda$ is 328, it follows that $\mathcal{H}_3$ may be defined over $\mathbb{Q}_p$. Let $f^{\rm adm}: Y^{\rm adm} \to X^{\rm adm}$ correspond to a cusp of label $5C$, and write $f': Y' \to X'$ (resp. $f'': Y'' \to X''$) for the corresponding three-point covers, in the notation of Section 6.3. The table above states that the decomposition group G' of Y' has order 55; it is no restriction to suppose that it is the Borel subgroup of G consisting of upper triangular matrices. The decomposition group G'' of Y'' is the full group $G = \mathrm{PSL}_2(p)$. By assumption $f'': Y'' \to X''$ is ramified of order 5 above μ. One checks that every cusp $[f^{\rm adm}]$ for which $(|G'|, |G''|, n) = (55, 660, 5)$ lies on the component $\mathcal{H}_3$.

To the cusp $[f^{\rm adm}]$ corresponds (noncanonically) a tuple $(g_0, g_1, g_2, g_3) \in \mathrm{Ni}(\mathbf{C})_3$ (more precisely, an orbit under the element $b_0 \in \mathcal{B}^{(4)}$). We may lift this tuple to a tuple $(h_0, h_1, h_2, h_3) \in C_1^2 \times C_2^2$ of elements in $\mathrm{SL}_2(p)$, with product $h_0h_1h_2h_3 = \pm 1$. Let $\tilde{h}_3 = \pm h_3$ be such that $h_0h_1h_2\tilde{h}_3 = 1$. We may lift f'' to an $\mathrm{SL}_2(p)$-Galois cover branched a three points which is branched at x_0, x_2, μ with class vector $\mathbf{C}'' = (C_1, C_2, C_3)$ where $C_3 = \mathcal{C}(l)$ for $l \in \{1, 2, 3, 4\}$. This cover corresponds to the Nielsen class $[(h_0, h_1h_2h_1^{-1}, h_1\tilde{h}_3]$. In fact, a more careful calculation of the cusps shows that $C_3 \in \{\mathcal{C}(1), \mathcal{C}(4)\}$. Lemma 6.4.1 states that f'' has bad reduction if and only if $C_3 = \mathcal{C}(4)$.

We may lift f' to a $\tilde{P}$-Galois cover with Nielsen class $[(h_1, \tilde{h}_3, (h_1\tilde{h}_3)^{-1})]$, where $\tilde{P} \subset \mathrm{SL}_2(p)$ has order 55 if $\tilde{h}_3 = h_3$ and order 110 otherwise. Write $(\mathbf{C}' = (C_1', C_2', C_3')$ for the class vector of f'. It follows that $C_1' = \mathcal{C}(4), C_2' \in \{\mathcal{C}(2), \mathcal{C}(3)\}, C_3' \in \{\mathcal{C}(1), \mathcal{C}(4)\}$ in the notation of Section 6.1. The cover $f': Y' \to X'$ factors as $Y' \to Z' \to X'$, where $\mathrm{Gal}(Y', Z') \simeq \mathbb{Z}/p$ and $g': Z' \to X'$ is cyclic of order $p-1 = 10$ or $(p-1)/2 = 5$. Renormalizing the branch points of g' to $x = 0, 1, \infty$, we may identify Z' with a connected component of the smooth projective curve given by the Kummer equation

$$z^{p-1} = x^{a_1}(x-1)^{a_2}, \quad 0 < a_0, a_1, a_3 < p-1, \quad a_0 + a_1 + a_2 \equiv 0 \bmod p-1.$$

The statement on the class vector of f' implies that

$$a_0 \in \{1, 4, 6, 9\}, \quad a_1 \in \{4, 6\}, \quad a_2 \in \{2, 3, 7, 8\}.$$

It follows that the only possibilities are

$$(a_0, a_1, a_2) \in \{(1, 1, 8), (6, 1, 3)\}.$$

In particular, $a_0 + a_1 + a_2 = p - 1$. But this implies that f' has bad reduction. This implies that all cusps of label $5C$ are bad cusps. By symmetry, the same follows for the cusps of label $5D$. This proves (a). Part (b) follows by a similar argument.

We conclude that the bad degree $d_{\rm bad}$ is greater than or equal to $4 \cdot 11 + 4 \cdot 1 + 2 \cdot 16 \cdot 5 = 208$. Let $\mathcal{B}$ be a bad horizontal component. Theorem 5.3.1 implies that the map $\mathcal{B} \to \mathbb{P}^1_\lambda$ is inseparable. This implies that p divides the bad degree. Since $208 \leq d_{\rm bad} \leq 208 + 6 \cdot 8$, we conclude that $d_{\rm bad} = 220$. This proves (c). □

It is not so easy to directly count the number of bad cusps of label $6A$, as we did for the cusps of label $5*$. The reason is that there are cusps of label $6A$ occurring also in the component $\mathcal{H}_2$, and it is more difficult, though probably nor impossible, to distinguish between the two.

Since the degree of π_2 is 2, the number N_1 of bad horizontal components, counted with multiplicities, whose underlying reduced subscheme is isomorphic to B_0^1 is equal to $(220 - 4 \cdot 2 \cdot 11)/11 = 12$.

Denote by $\mathbb{H}_3 \to \mathbb{P}^1_\lambda$ the Galois closure of ϖ_3. As we remarked before, its Galois group Γ_3 is isomorphic to A_{82}. The calculation of the bad horizontal components clearly gives some information on the reduction of $\mathbb{H}_3 \to \mathbb{P}^1_\lambda$. Since the bad degree is nonzero and strictly less than the degree of ϖ_3, it follows that $\mathbb{H}_3 \to \mathbb{P}^1_\lambda$ has bad reduction. Since the order of Γ_3 is strictly less $p^2 = 121$, the order of the inertia group I_0 of an irreducible component of $\bar{\mathbb{H}}_3$ above the original component is an elementary abelian p-group. One can limit the possibilities for the order of this inertia group from the bad degree. It should be possible to get more information by using a more careful analyses of the universal deformation rings and making a more systematic study of the Swan conductors of G-Galois covers such that the Sylow p-subgroup of G is elementary abelian.

References

[1] Y. André. *Period mappings and differential equations. From $\mathbb{C}$ to $\mathbb{C}_p$.* Number 12 in MSJ Memoirs. Mathematical Society of Japan, 2003.

[2] A. Beauville. Les familles stables de courbes elliptiques sur $\mathbf{P}^1$ admettant quatre fibres singulières. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(19):657–660, 1982.

[3] P. Berthelot and A. Ogus. *Notes on crystalline cohomology.* Princeton University Press, 1978.

[4] F. Beukers. On Dwork's accessory parameter problem. *Math. Z.*, 241:425–444, 2002.

[5] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models.* Number 21 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1990.

[6] I. I. Bouw. The p-rank of curves and covers of curves. In *Courbes semi-stables et groupe fondamental en géométrie algébrique (Luminy, 1998)*, volume 187 of *Progr. Math.*, pages 267–277. Birkhäuser, 2000.

[7] I. I. Bouw. The p-rank of ramified covers of curves. *Compositio Math.*, 126:295–322, 2001.

[8] I. I. Bouw. Reduction of the Hurwitz space of metacyclic covers. *Duke Math. J.*, 121:75–111, 2004.

[9] I. I. Bouw and R. J. Pries. Rigidity, reduction, and ramification. *Math. Ann.*, 326:803–824, 2003.

[10] I. I. Bouw and S. Wewers. Indigenous bundles with nilpotent p-curvature. In preparation.

[11] I. I. Bouw and S. Wewers. The local lifting problem for dihedral groups. math.AG/0409395.

[12] I. I. Bouw and S. Wewers. Reduction of covers and Hurwitz spaces. *J. Reine Angew. Math.*, 574:1–49, 2004.

[13] I. I. Bouw and S. Wewers. Stable reduction of modular curves. In *Modular curves and abelian varieties*, volume 224 of *Progress in Math.*, pages 1–22. Birkhäuser Verlag, 2004.

[14] P. Deligne and L. Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89:247–270, 1987.

[15] B. M. Dwork. *Lectures on p-adic differential equations*, volume 253 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, New York, 1982. With an appendix by Alan Adolphson.

[16] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley-Interscience, 1978.

[17] T. Honda. Algebraic differential equations. In *Symposia Mathematica, Vol. XXIV (Sympos., INDAM, Rome, 1979)*, pages 169–204. Academic Press, London, 1981.

[18] O. Hyodo. Wild ramification in the imperfect residue field case. In *Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986)*, volume 12 of *Adv. Stud. Pure Math.*, pages 287–314. North-Holland, Amsterdam, 1987.

[19] Y. Ihara. On the differentials associated to congruence relations and the Schwarzian equations defining uniformizations. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 21:309–332, 1974.

[20] L. Illusie. Frobenius and Hodge degeneration. In *Introduction to Hodge theory*, volume 8 of *SMF/AMS texts and monographs*, pages 99–149. Soc. Math. France, Paris, 2002.

[21] M. Kaneko and D. Zagier. Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 97–126. Amer. Math. Soc., 1998.

[22] K. Kato. Swan conductors with differential values. In *Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986)*, volume 12 of *Adv. Stud. Pure Math.*, pages 315–342. North-Holland, Amsterdam, 1987.

[23] N. M. Katz. On the differential equations satisfied by period matrices. *Inst. Hautes Études Sci. Publ. Math.*, 35:223–258, 1968.

[24] N. M. Katz. Nilpotent connections and the monodromy theorem: applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, 39:355–412, 1970.

[25] N. M. Katz. Travaux de Dwork. In *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 409*, pages 167–200. Lecture Notes in Math., Vol. 317. Springer, 1973.

[26] N. M. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Inst. Fundamental Res., Bombay, 1981.

[27] N. M. Katz. Expansion-coefficients as approximate solution of differential equations. *Astérisque*, 119-120:6, 183–189, 1984.

[28] N. M. Katz. Internal reconstruction of unit-root F-crystals via expansion-coefficients. *Ann. Sci. École Norm. Sup. (4)*, 18(2):245–285, 1985. With an appendix by Luc Illusie.

[29] H. Kraft. Kommutative algebraische Gruppen (mit Anwendungen auf p-divisibele Gruppen und abelsche Varietäten). manuscript, Universität Bonn, 1975.

[30] M. Matignon. Vers un algorithme pour la réduction stable des revêtements p-cycliques de la droite projective sur un corps p-adique. *Math. Ann.*, 325(2):323–354, 2003.

[31] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[32] B. Mazur. Frobenius and the Hodge filtration. *Bull. Amer. Math. Soc.*, 78:653–667, 1972.

[33] B. Mazur. Frobenius and the Hodge filtration (estimates). *Ann. of Math. (2)*, 98:58–95, 1973.

[34] J. S. Milne. *Étale Cohomology.* Princeton Univ. Press, 1980.

[35] S. Mochizuki. A theory of ordinary p-adic curves. *Publ. RIMS, Kyoto Univ.*, 32:957–1151, 1996.

[36] S. Mochizuki. *Foundations of p-adic Teichmüller theory.* Number 11 in Studies in Advanced Math. AMS/IP, 1999.

[37] D. Mumford. *Abelian varieties.* Oxford Univ. Press, 1970.

[38] A. Ogus. Elliptic crystals and modular motives. *Adv. Math.*, 162:173–216, 2001.

[39] L. Ramero. Local monodromy in non-archimedean analytic geometry. fifth release, math.AG/0310418.

[40] M. Raynaud. Spécialisation des revêtements en caractéristique $p > 0$. *Ann. Scient. Éc. Norm. Sup.*, pages 87–126, 1999.

[41] J.-P. Serre. *Corps locaux.* Hermann, Paris, 1968. Deuxième édition, Publications de l'Université de Nancago, No. VIII.

[42] J.-P. Serre. *Groupes algébriques et corps de classes.* Publications de l'Institut Mathématique de l'Université de Nancago, 7. Hermann, Paris, 1984.

[43] J.-P. Serre. *Topics in Galois Theory.* Research notes in mathematics. Jones and Bartlett Publishers, 1992.

[44] J. Stienstra and F. Beukers. On the Picard-Fuchs equation and the formal Brauer group of certain elliptic $K3$-surfaces. *Math. Ann.*, 271(2):269–304, 1985.

[45] J. Stienstra, M. van der Put, and B. van der Marel. On p-adic monodromy. *Math. Z.*, 208:309–325, 1991.

[46] H. A. Verrill. Picard-Fuchs equations of some families of elliptic curves. In *Proceedings on Moonshine and related topics (Montréal, QC, 1999)*, volume 30 of *CRM Proc. Lecture Notes*, pages 253–268. Amer. Math. Soc., 2001.

[47] H. Völklein. *Groups as Galois Groups.* Number 53 in Cambridge Studies in Adv. Math. Cambridge Univ. Press, 1996.

[48] S. Wewers. Construction of Hurwitz spaces. Thesis, Preprint No. 21 of the IEM, Essen, 1998.

[49] S. Wewers. Reduction and lifting of special metacyclic covers. *Ann. Scient. Éc. Norm. Sup.*, 36:113–138, 2003.

[50] S. Wewers. Three point covers with bad reduction. *J. Amer. Math. Soc.*, 16(4):991–1032, 2003.

[51] S. Wewers. Formal deformation of curves with group scheme action. *Ann. Inst. Fourier*, 55:???, 2005.

[52] M. Yoshida. *Fuchsian differential equations.* Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, Braunschweig, 1987.