Mean geometry of 2D random fields

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Joint works

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Outlines

Lecture 1 :

Introduction to random fields

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and $d \geq 2$

Definition

A (real) random field indexed by $\mathbb{R}^d$ is just a collection of real random variables $X(x) : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable, $\forall x \in \mathbb{R}^d$.

Exple : $d = 2$, $X(x)(\omega) =$ grey level of a picture at point x. In practice data are only available on pixels $S = \{0, 1, \ldots, n-1\}^2 \subset \mathbb{R}^2$ for an image of size $n \times n$.

Introduction to random fields

Definition

The distribution of $(X(x))_{x\in\mathbb{R}^d}$ *is given by all its finite dimensional distribution (fdd) ie the distribution of all real random vectors*

$$(X(x_1)\dots,X(x_k)) \text{ for } k\geq 1, x_1,\dots,x_k\in\mathbb{R}^d.$$

Joint distributions are often difficult to compute !

Definition

$(X(x))_{x\in T}$ *is a second order field if* $\mathbb{E}(X(x)^2)<+\infty$ *for all* $x\in\mathbb{R}^d$*.*

- *Mean function* $m_X : x\in\mathbb{R}^d\to\mathbb{E}(X(x))\in\mathbb{R}$
- *Covariance function* $K_X : (x,y)\in\mathbb{R}^d\times\mathbb{R}^d\to Cov(X(x),X(y))\in\mathbb{R}.$

Introduction to random fields

When $m_X = 0$, the field X is centered. Otherwize $Y = X - m_X$ is centered and $K_Y = K_X$.

Proposition

A function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a covariance function iff*

1. K *is symmetric*
2. K *is positive definite :* $\forall k \geq 1, x_1, \ldots, x_k \in \mathbb{R}^d, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$,

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j K(x_i, x_j) \geq 0.$$

Gaussian fields

Definition

$(X(x))_{x\in\mathbb{R}^d}$ is a Gaussian field if $\forall k \geq 1, x_1, \ldots, x_k \in \mathbb{R}^d$

$$(X(x_1), \ldots, X(x_k)) \text{ is a Gaussian vector of } \mathbb{R}^k,$$

EQ $\forall \lambda_1, \ldots, \lambda_k \in \mathbb{R}$, the real random variable $\displaystyle\sum_{i=1}^{k} \lambda_i X(x_i)$ is a Gaussian variable.

Proposition

When $(X(x))_{x\in\mathbb{R}^d}$ is Gaussian, $(X(x))_{x\in\mathbb{R}^d}$ is a second order field and its law is determined by its mean function $m_X : x \mapsto \mathbb{E}(X(x))$ and its covariance function $K_X : (x, y) \mapsto Cov(X(x), X(y))$.

Theorem (Komogorov)

Let $m : \mathbb{R}^d \to \mathbb{R}$ and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, symmetric and positive definite, then there exists a Gaussian field with mean m and covariance K.

Stationarity

Definition

$X = (X(x))_{x\in\mathbb{R}^d}$ (strongly) stationary if, $\forall x_0 \in \mathbb{R}^d$, $(X(x+x_0))_{x\in\mathbb{R}^d}$ has the same law than X.

Proposition

If $X = (X(x))_{x\in\mathbb{R}^d}$ is stationary and second order, $\forall x_0 \in \mathbb{R}^d$,

- $m_X(x) = m_X$
- $K_X(x,y) = \rho_X(x-y)$ *with* $\rho_X : \mathbb{R}^d \to \mathbb{R}$ *even s.t.*
  1. $\rho_X(0) \geq 0$
  2. $|\rho_X(x)| \leq \rho_X(0)\ \forall x \in \mathbb{R}^d$
  3. ρ_X *is of positive type ie*
     $\forall k \geq 1, x_1, \ldots, x_k \in \mathbb{R}^d, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$,

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \rho_X(x_i - x_j) \geq 0.$$

Stationarity

Theorem (Bochner 1932)

An even continuous function $\rho : \mathbb{R}^d \to \mathbb{R}$ is of positive type if and only if $\rho(0) > 0$ and there exists a symmetric probability measure μ on $\mathbb{R}^d$ such that

$$\rho(x) = \rho(0) \int_{\mathbb{R}^d} e^{i\xi \cdot x} d\mu(\xi).$$

In other words there exists a symmetric random vector Z on $\mathbb{R}^d$ such that

$$\rho(x) = \rho(0)\mathbb{E}(e^{ix\cdot Z}).$$

Rk : When ρ is the covariance of the stationary field X, μ is called the spectral measure of X.

Definition

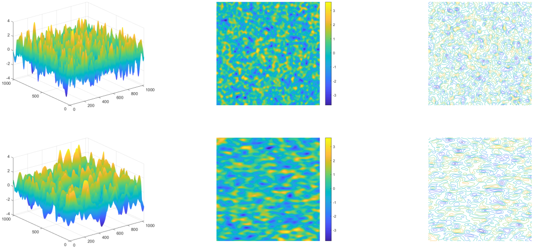
The field is **standard** *if it is centered ($m = 0$) and unit variance ($\rho(0) = 1$).*

Isotropy

Definition

$X = (X(x))_{x\in\mathbb{R}^d}$ isotropic if, $\forall Q$ rotation, $(X(Qx))_{x\in\mathbb{R}^d}$ has the same law than X.

Rk : A stationary Gaussian random field is isotropic iff $\forall Q$, $\rho(Qx) = \rho(x)$
Exple : $d = 2$ $\rho(x) = \exp(-\frac{\gamma_1}{2}x_1^2)\exp(-\frac{\gamma_2}{2}x_2^2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$ and $\gamma_1, \gamma_2 \in (0, +\infty)$: X isotropic iff $\gamma_1 = \gamma_2$



view 3D view 2D some level lines

$n = 2^{10}$; top $\gamma_2 = \gamma_1 = 0.005$; bottom $\gamma_1 = 0.001$

Smooth Gaussian stationary random fields

Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be an even $C^{2k+\varepsilon}$ function (for instance) of positive type. Then one can find $(\Omega, \mathcal{A}, \mathbb{P})$ a complete probability space and

$$X : \Omega \times \mathbb{R}^d \to \mathbb{R}$$

such that X is a **centered Gaussian stationary C^k random field** :

- $\forall \omega \in \Omega$, $x \in \mathbb{R}^d \mapsto X(\omega, x) \in \mathbb{R}$ is C^k ;
- $\left(\partial_{\mathrm{j}}^{|\mathrm{j}|} X\right)_{|\mathrm{j}| \leq k}$ jointly Gaussian stationary with

$$\mathrm{Cov}\left(\partial_{\mathrm{j}}^{|\mathrm{j}|} X(x), \partial_{\mathrm{j}'}^{|\mathrm{j}'|} X(0)\right) = (-1)^{|\mathrm{j}'|} \partial_{\mathrm{j}+\mathrm{j}'}^{|\mathrm{j}|+|\mathrm{j}'|} \rho(x).$$

In particular $\Gamma_{\nabla X} := (-\partial_{ij}\rho(0))_{1 \leq i,j \leq d} = -D^2\rho(0)$ is the covariance of the centered Gaussian vector $\nabla X(x) = (\partial_1 X(x), \ldots, \partial_d X(x))$. When X is also isotropic $\Gamma_{\nabla X} = \gamma_2 I_2$ with γ_2 the second spectral moment
Rk : Any stationary C^1 Gaussian random field Y may be written as

$$Y = m + \sigma X \circ Q,$$

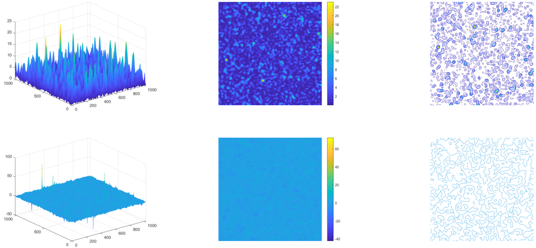
with X standard with $\rho_X = \frac{1}{\sigma^2}\rho_Y$ and $\Gamma_{\nabla X} = \frac{1}{\sigma^2} Q \Gamma_{\nabla Y} Q^T$.

Smooth Gaussian type random fields

A random field X is of **Gaussian type** if $X = F(\mathrm{G})$: $F : \mathbb{R}^\ell \to \mathbb{R}$ and $\mathrm{G} = (G_1, \ldots, G_\ell)$ is a family of *i.i.d.* Gaussian standard stationary random fields

Examples

- $\chi^2(k)$ random field : $F : x \in \mathbb{R}^k \to \|x\|^2$
- Student(k) : $x = (z, \mathrm{y}) \in \mathbb{R} \times \mathbb{R}^k \mapsto F(x) := z / \sqrt{\|\mathrm{y}\|^2 / k}$



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$n = 2^{10}$, $\gamma_2 = \gamma_1 = 0.005$; top $\chi^2(2)$ bottom Student(4)

Shot noise random fields

A **(Poisson) shot noise random field** is a random function $X : \mathbb{R}^d \to \mathbb{R}$ given by

$$\forall x \in \mathbb{R}^d, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i), \text{ where}$$

- $\{x_i\}_{i \in I}$ is a Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^d$,
- $\{m_i\}_{i \in I}$ are independent « marks » with distribution $F(dm)$ on $\mathbb{R}^\ell$, and independent of $\{x_i\}_{i \in I}$.
- The functions g_m are real-valued deterministic functions, called **spot functions**, such that

$$\int_{\mathbb{R}^\ell} \int_{\mathbb{R}^d} |g_m(y)| \, dy \, F(dm) < +\infty.$$

Here we mainly consider, for sake of simplicity, $\ell = 1$ with a single $L^1(\mathbb{R}^d)$ function g randomly weighted or dilated : $M \sim F$ is a probability measure on $\mathbb{R}$ or $(0, +\infty)$ and

$$g_m(y) = mg(y) \text{ or } g_m(y) = g(y/m).$$

Shot noise random fields

It is therefore a stationary and integrable field with

$$\mathbb{E}(X(x)) = \lambda \int_{\mathbb{R}^\ell} \int_{\mathbb{R}^d} g_m(y)\, dy\, F(dm).$$

The characteristic function of $X(x)$ is given by

$$\mathbb{E}\left(e^{itX(x)}\right) = \exp\left(\lambda \int_{\mathbb{R}^\ell \times \mathbb{R}^d} [e^{i[tg_m(y)]} - 1] F(dm) dy\right).$$

When g is smooth and $|\mathrm{j}| \leq k+1$

$$\int_{\mathbb{R}^\ell} \int_{\mathbb{R}^d} |\partial_{\mathrm{j}}^{|\mathrm{j}|} g_m(y)|\, dy\, F(dm) < +\infty$$

X is C^k and we have also access to joint law of $(\partial_{\mathrm{j}} X(x))_{x \in \mathbb{R}^d, |\mathrm{j}| \leq k}$ via characteristic function. In particular the joint characteristic function of $X(x)$ and $\partial_j X(x)$ is

$$\begin{aligned} \varphi(t,s) &= \mathbb{E}\left(e^{itX(x)+is\partial_j X(x)}\right) \\ &= \exp\left(\lambda \iint [e^{itg_m(y)+is\partial_j g_m(y)} - 1]\, F(dm)\, dy\right) \end{aligned}$$

Statistical properties of shot noise random fields

- If $\int_{\mathbb{R}^\ell}\int_{\mathbb{R}^d} g_m(y)^2 dy F(dm) < +\infty$, then X has second-order moments

$$\begin{aligned}
\mathrm{Cov}(X(z), X(z+x)) &= \mathbb{E}(X(z)X(z+x)) - \mathbb{E}(X(z))\mathbb{E}(X(z+x)) \\
&= \lambda \int_{\mathbb{R}^\ell}\int_{\mathbb{R}^d} g_m(y) g_m(y-x)\, dy\, F(dm) \\
&= \lambda \rho(x).
\end{aligned}$$

In particular

$$\mathrm{Var}(X(x)) = \mathrm{Var}(X(0)) = \lambda \int_{\mathbb{R}^\ell}\int_{\mathbb{R}^d} g_m(y)^2\, dy\, F(dm).$$

- When moreover the intensity λ goes to $+\infty$, the normalized random field

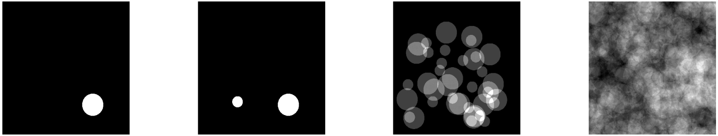
$$Z(x) = \frac{X(x) - \mathbb{E}(X(x))}{\sqrt{\lambda}}$$

converges (f.d.d.) to a stationary centered Gaussian field with covariance ρ.

Example 1 : disk with random radius

Let $d = 2$, $g = 1_D$, $U = (0, T)^2$ and consider random disk of radius $m = m_1$ or $m = m_2$ with $0 < m_1 < m_2$ (each with probability $1/2$) with intensity $\lambda > 0$

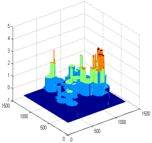
- The number of centers in $(-m_2, T + m_2)^2$ is a Poisson random variable of parameter $\lambda(T + 2m_2)^2 \longrightarrow n$
- The centers $x_1, \dots, x_n$ are thrown uniformly, independently on $(-m_2, T + m_2)^2$
- The radius $R_1, \dots, R_n$ are attached to each center by flipping a coin to choose between m_1 or m_2.



Excursion set

We consider the excursion set or the level set of level $t \in \mathbb{R}$ of X in U defined by

$$E_X(t) \cap U := \{x \in U; X(x) \geq t\} \text{ with } E_X(t) = \{X \geq t\}.$$



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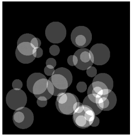

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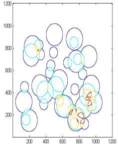
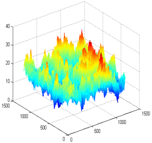
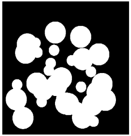
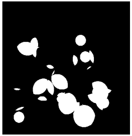


some level lines



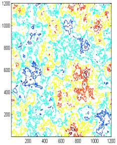


$t = 0.5$



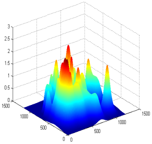
$t = 1.5$

$t = 2.5$

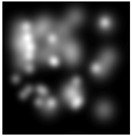


Example 2 : Gaussian kernel

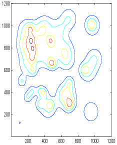
Let us choose $g(x) = e^{-\frac{\|x\|^2}{2}}$ instead of 1_D.



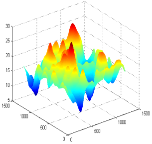
view 3D

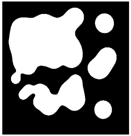


view 2D



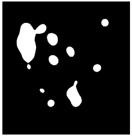
some level lines



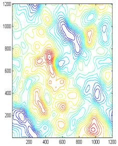


$t = 0.5$

$t = 1$



$t = 1.5$



Main questions

What can be said about "mean" geometry of excursion sets ? Area ? Perimeter ? Euler Characteristic=# connected components – # holes ?

Known results for

- Boolean model : Mecke (2001), Mecke, Wagner (1991)
- Smooth Gaussian random fields : Adler (2000), Adler, Taylor (2007), Azaïs, Wschebor (2009), ...
- High levels : Adler, Samorodnitsky, Taylor (2010,2013),...

Two different frameworks

1. Elementary : g is piecewize constant with compact support
2. Smooth : g is at least C^3

Outlines

Curvature measures

Let $E \subset \mathbb{R}^2$ be a "nice set". Its curvature measures $\Phi_j(E, \cdot)$, for $j = 0, 1, 2$, are defined for any Borel set $U \subset \mathbb{R}^2$ by

- $\Phi_2(E, U) = |E \cap U|$, occupied area
- $\Phi_1(E, U) = \frac{1}{2}\mathcal{H}^1(\partial E \cap U) = \frac{1}{2}\mathrm{Per}(E, U)$, regularity property
- $\Phi_0(E, U) = \frac{1}{2\pi}\mathrm{TC}(\partial E, U)$, connectivity property

where $\mathcal{H}^1(\partial E \cap U)$ is the lenght and $\mathrm{TC}(\partial E, U)$ the total curvature of the positively oriented curve ∂E in U.

For E a compact or convex set and $E \subset U$ also related to Minkowski or intrinsic volumes, widely used in mathematical morphology, convex and integral geometry : Hadwiger (1957), Federer (1959), Santaló (1976), Schneider & Weil (2008),...

Piecewise regular curve

A Jordan curve $\Gamma \subset \mathbb{R}^2$ is piecewise regular if $\Gamma = \mathcal{R}_\Gamma \cup \mathcal{C}_\Gamma$ with $\#\mathcal{C}_\Gamma < +\infty$

- for $x \in \mathcal{R}_\Gamma$ one has $x = \gamma(0)$ for some $s \in (-\varepsilon, \varepsilon)$ with $\gamma : (-\varepsilon, \varepsilon) \to \Gamma$ C^2, arc length parametrized. Then, $\mathcal{H}^1(\gamma(-\varepsilon, \varepsilon)) = 2\varepsilon$, $\nu_\Gamma(x) = \gamma'(0)^\perp$.
  The **signed curvature** $\kappa_\Gamma(x)$ of Γ at x is

$$\kappa_\Gamma(x) = \langle \gamma''(0), \gamma'(0)^\perp \rangle.$$

- for $x \in \mathcal{C}_\Gamma$ one has $x = \gamma(0)$ with $\gamma : (-\varepsilon, \varepsilon) \to \Gamma$ continuous and C^2 on $(-\varepsilon, \varepsilon) \setminus \{0\}$ s.t. γ' admits limits $\gamma'(0^-) \in S^1$ and $\gamma'(0^+) \in S^1$ at 0. Then, $\mathcal{H}^1(\gamma(-\varepsilon, \varepsilon)) = 2\varepsilon$.
  The **turning angle** at a corner point $x = \gamma(0) \in \mathcal{C}_\Gamma$ is the angle $\alpha_\Gamma(x) \in (-\pi, \pi)$ between the tangent "before" and the one "after" x

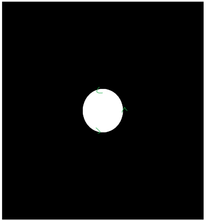
$$\alpha_\Gamma(x) = \operatorname{Arg} \gamma'(0^+) - \operatorname{Arg} \gamma'(0^-) \qquad \in (-\pi, \pi).$$

Total curvature

The **total curvature** of Γ in U is defined as

$$\mathrm{TC}(\Gamma, U) := \int_{\mathcal{R}_\Gamma \cap U} \kappa_\Gamma(x)\mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} \alpha_\Gamma(x).$$

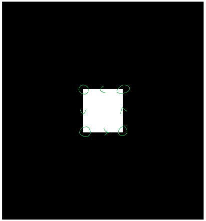
$D = \{x; \|x\| \leq R\}$



$\kappa_{\partial D}(x) = 1/R$ & $\mathcal{C}_{\partial D}(x) = \emptyset$

$\mathrm{TC}(\partial D, U) = 1/R \times 2\pi R = 2\pi$

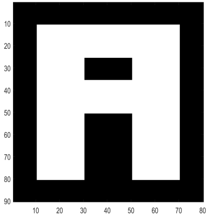
$C = \{x; \|x\|_\infty \leq R\}$



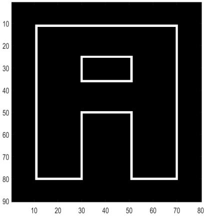
$\kappa_{\partial C}(x) = 0$ & $\alpha_{\partial D}(x) = \pi/2$

$\mathrm{TC}(\partial C, U) = 4 \times \pi/2 = 2\pi$

Total curvature : example



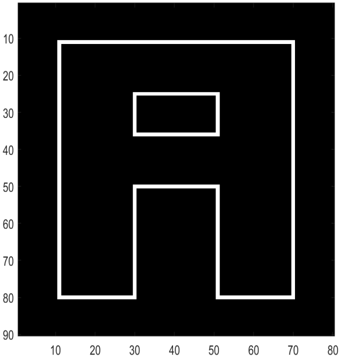
A



$\partial A = \Gamma_1 \cup \Gamma_2$

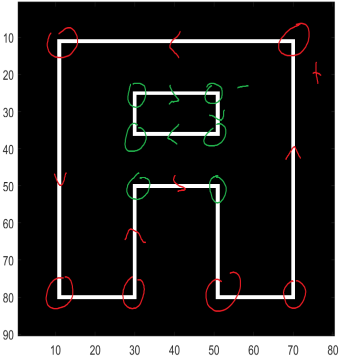
- $\forall x \in \mathcal{R}_{\partial A}$, $\kappa_{\partial A}(x) = 0$
- $\#\mathcal{C}_{\partial A} = 12$ and $\alpha_{\partial A}(x) = \pm\pi/2$

Total curvature : example



?

Total curvature : example



$$
\begin{aligned}
\mathrm{TC}(\partial A, U) &= \mathrm{TC}(\Gamma_1, U) + \mathrm{TC}(\Gamma_2, U) \\
&= (6 \times \pi/2 + 2 \times (-\pi/2)) + 4 \times (-\pi/2) = 0 \\
\mathrm{Per}(A, U) &= 380
\end{aligned}
$$

Total curvature and Euler characteristic

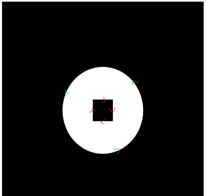
Theorem (Gauss-Bonnet)

Let $E \subset U$ be a regular region ie $E = \overline{\overset{o}{E}}$ such that $\partial E = \cup_{i=1}^{n} \Gamma_i$ is a finite union of disjoint positively oriented Jordan piecewise regular curves. then

$$\mathrm{TC}(\partial E, U) := \sum_{i=1}^{n} \mathrm{TC}(\Gamma_i, U) = 2\pi\chi(E) \ \left(= \ 2\pi\Phi_0(E, U)\right),$$

where $\chi(E) \in \mathbb{Z}$ is the **Euler characteristic** *of E.*

$$\chi(E) = \#\text{connected components} - \#\ \text{holes}.$$



Geometry of excursion sets

Let $X = (X(x))_{x\in\mathbb{R}^2}$ be a stationary "nice" random field and $U \subset \mathbb{R}^2$ a bounded open rectangle. For $t \in \mathbb{R}$, we consider for the excursion set of level t

$$E_X(t) := \{x \in \mathbb{R}^2; X(x) \geq t\}.$$

the **LK curvatures** of the excursion set $E_X(t)$ within $\overline{U}$ are classicaly defined as

$$C_j(X,t,\overline{U}) := \Phi_j(E_X(t)\cap \overline{U}, \overline{U}), \quad \text{for } j = 0,1,2.$$

and, assuming the limits exist, the associated **LK densities** are

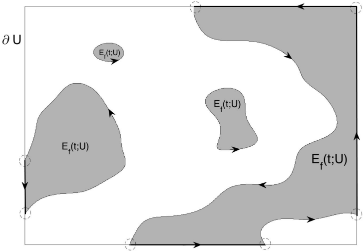
$$C_j^*(X,t) := \lim_{\overline{U}\nearrow\mathbb{R}^2} \frac{\mathbb{E}[C_j(X,t,\overline{U})]}{|\overline{U}|}, \quad \text{for } j = 0,1,2,$$

where $\lim\limits_{\overline{U}\nearrow\mathbb{R}^2}$ stands for the limit along any sequence of bounded rectangles that grows to $\mathbb{R}^2$. Note that for $j = 2$, by stationarity,

$$\mathbb{E}[C_2(X,t,\overline{U})] = |E_X(t)\cap\overline{U}| = \mathbb{P}(X(0)\geq t)|\overline{U}|, \text{ s.t.}$$

$$C_2^*(X,t) = \mathbb{P}(X(0) \geq t)$$

LK densities



Moreover

$$C_j(X,t,\overline{U}) = \Phi_j(E_X(t)\cap\overline{U},\overline{U}) = \Phi_j(E_X(t),U) + \Phi_j(E_X(t)\cap\overline{U},\partial U).$$

Then $\frac{\Phi_j(E_X(t),U)}{|U|}$ is an unbiased, strongly consistent estimator of $C_j^*(X,t)$

$$\frac{\mathbb{E}\left[\Phi_j(E_X(t),U)\right]}{|U|} = C_j^*(X,t) \text{ with } \lim_{U\nearrow\mathbb{R}^2}\frac{\Phi_j(E_X(t),U)}{|U|} = C_j^*(X,t) \text{ a.s..}$$

Weak framework

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a "nice" function and note

$$E_f(t) = \{x \in \mathbb{R}^2; f(x) \geq t\}.$$

Remark that when $t < \min_{\overline{U}} f$ or $t > \max_{\overline{U}} f$, then $\partial E_f(t) \cap \overline{U} = \emptyset$. We consider for any h bounded continuous function on $\mathbb{R}$

- $$L_f \Phi_1(h, U) := \int_{\mathbb{R}} h(t)\Phi_1(E_f(t), U)dt = \frac{1}{2}\int_{\mathbb{R}} h(t)\mathcal{H}^1(\partial E_f(t) \cap U)dt;$$

- $$L_f \Phi_0(h, U) := \int_{\mathbb{R}} h(t)\Phi_0(E_f(t), U)dt = \frac{1}{2\pi}\int_{\mathbb{R}} h(t)\mathrm{TC}(\partial E_f(t), U)dt.$$

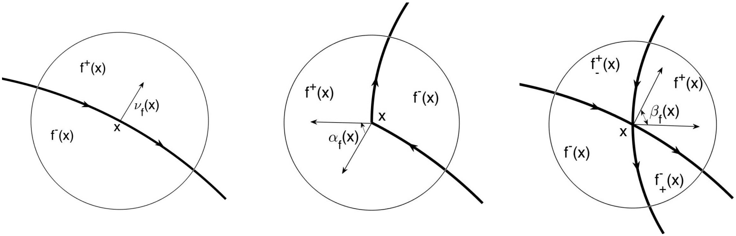
Outlines

The case of elementary functions

Definition

The function f is said elementary function if f is a piecewise constant function taking a locally finite number of values and if $\mathcal{S}_f$ the discontinuity set of f can be decomposed as

$$\mathcal{S}_f = \bigcup_{t\in\mathbb{R}} \partial E_f(t) = \mathcal{R}_f \cup \mathcal{C}_f \cup \mathcal{I}_f, \text{ where :}$$



$x \in \mathcal{R}_f$ $\qquad$ $x \in \mathcal{C}_f$ $\qquad$ $x \in \mathcal{I}_f$

From left to right : a regular point, a corner point and an intersection point

The case of elementary functions

In this case writing $H(t) = \int_0^t h(s)ds$ we obtain

$$L_f\Phi_1(h,U) = \frac{1}{2}\int_{\mathcal{R}_f\cap U}[H(f^+(x)) - H(f^-(x))]\mathcal{H}^1(dx)$$

$$\begin{aligned}L_f\Phi_0(h,U) &= \frac{1}{2\pi}\int_{\mathcal{R}_f\cap U}[H(f^+(x)) - H(f^-(x))]\kappa_f(x)\mathcal{H}^1(dx)\\ &+ \frac{1}{2\pi}\sum_{x\in\mathcal{C}_f\cap U}[H(f^+(x)) - H(f^-(x))]\alpha_f(x)\\ &+ \frac{1}{2\pi}\sum_{x\in\mathcal{I}_f\cap U}[H(f^+(x)) + H(f^-(x)) - H(f_-^+(x)) - H(f_+^-(x))]\beta_f(x).\end{aligned}$$

Elementary Shot Noise fields

Consider

$$X(x) = \sum_i g_{m_i}(x - x_i)$$

For g_m elementary with $\mathcal{S}_{g_m} = \mathcal{R}_{g_m} \cup \mathcal{C}_{g_m}$,+ technical assumptions, the shot noise field X is elementary with

- $\mathcal{R}_X = \cup_i \tau_{x_i} \mathcal{R}_{g_{m_i}} \setminus \left(\cup_{i \neq j} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \right)$, and for $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \mathcal{R}_X$
  $X^{\pm}(x) = \sum_{j \neq i} g_{m_j}(x - x_j) + g^{\pm}_{m_i}(x - x_i)$
- $\mathcal{C}_X = \cup_i \tau_{x_i} \mathcal{C}_{g_{m_i}}$, and for $x \in \tau_{x_i} \mathcal{C}_{g_{m_i}} \cap \mathcal{R}_X$
  $X^{\pm}(x) = \sum_{j \neq i} g_{m_j}(x - x_j) + g^{\pm}_{m_i}(x - x_i)$
- $\mathcal{I}_X = \left(\cup_{i \neq j} \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}} \right)$ and for $x \in \tau_{x_i} \mathcal{R}_{g_{m_i}} \cap \tau_{x_j} \mathcal{R}_{g_{m_j}}$
  $X^{\pm}(x) = \sum_{k \neq i,j} g_{m_k}(x - x_k) + g^{\pm}_{m_i}(x - x_i) + g^{\pm}_{m_j}(x - x_j)$ and
  $X^{+}_{-}(x) = \sum_{k \neq i,j} g_{m_k}(x - x_k) + g^{+}_{m_i}(x - x_i) + g^{-}_{m_j}(x - x_j)$
  $X^{-}_{+}(x) = \sum_{k \neq i,j} g_{m_k}(x - x_k) + g^{-}_{m_i}(x - x_i) + g^{+}_{m_j}(x - x_j)$

Weak formula for Shot Noise fields

Using Slivnyak-Mecke formula, Fubini and stationarity

$$\begin{aligned}
\mathbb{E}(L_X\Phi_1(h,U)) &= \lambda|U|\int_{\mathbb{R}^\ell} L_{g_m}\Phi_1(\overline{h}_{X_\Phi(0)},\mathbb{R}^2)F(dm)\\
\mathbb{E}(L_X\Phi_0(h,U)) &= \lambda|U|\int_{\mathbb{R}^\ell} \left(L_{g_m}\Phi_0(\overline{h}_{X_\Phi(0)},\mathbb{R}^2)+\lambda I(\overline{h}_{X_\Phi(0)},m)\right)F(dm),
\end{aligned}$$

where

$$\overline{h}_{X_\Phi(0)}(s)=\mathbb{E}(h(X_\Phi(0)+s)),$$

and

$$I(\overline{h}_{X_\Phi(0)},m)=\frac{1}{4\pi}\int_{\mathbb{R}^\ell}\int_{\mathbb{R}^2}\sum_{z\in\tau_x\mathcal{R}_{g_m}\cap\mathcal{R}_{g_{m'}}} d_{S^1}(\nu_{g_m}(z-x),\nu_{g_{m'}}(z))$$

$$\times\int_{g^-_{m'}(z)}^{g^+_{m'}(z)}\left(\overline{h}_{X_\Phi(0)}(s+g^+_m(z-x))-\overline{h}_{X_\Phi(0)}(s+g^-_m(z-x))\right)dsdxF(dm').$$

LK densities for Shot Noise fields

Considering $g = 1_D$ and $g_m = 1_{D_m}$ so that

- $\Phi_1(E_{g_m}(t), \mathbb{R}^2) = \frac{1}{2}\mathrm{Per}(E_{g_m}(t), \mathbb{R}^2) = \pi m 1_{0<t\leq 1}$,
- $\Phi_0(E_{g_m}(t), \mathbb{R}^2) = 1_{0<t\leq 1}$

and according to the kinematic formula we have

$$\int_{\mathbb{R}^2} \sum_{z\in\tau_x\partial D_m\cap\partial D_{m'}} d_{S^1}(\nu_{g_m}(z-x), \nu_{g_{m'}}(z))dx = 2\pi m \times 2\pi m'.$$

Introducing $\overline{p} = \int_{\mathbb{R}^+} 2\pi m F(dm)$

$$\begin{aligned}
\mathbb{E}(L_X\Phi_1(h,U)) &= \frac{\lambda}{2}|U| \int_0^1 \overline{h}_{X_\Phi(0)}(s)\overline{p}ds \\
\mathbb{E}(L_X\Phi_0(h,U)) &= \lambda|U| \int_0^1 \left(\overline{h}_{X_\Phi(0)}(s) + \frac{\lambda}{4\pi}\overline{p}^2\left(\overline{h}_{X_\Phi(0)}(s+1) - \overline{h}_{X_\Phi(0)}(s)\right)\right) ds.
\end{aligned}$$

Application to Shot Noise

Since $X_\Phi(0) \sim \mathcal{P}(\lambda\overline{a})$ with $\overline{a} = \int_{\mathbb{R}^+} \pi m^2 F(dm)$, we get for $s \in [0,1)$

$$\begin{aligned}
\overline{h}_{X_\Phi(0)}(s) &= \mathbb{E}(h(X_\Phi(0)+s)) \\
&= \sum_{k=0}^{+\infty} h(k+s) e^{-\lambda\overline{a}} \frac{(\lambda\overline{a})^k}{k!} = \sum_{k=0}^{+\infty} h(k+s) e^{-\lambda\overline{a}} \frac{(\lambda\overline{a})^{\lfloor k+s \rfloor}}{\lfloor k+s \rfloor !}
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}} h(t) C_1^*(X,t)\,dt &= \frac{\mathbb{E}(L_X \Phi_1(h,U))}{|U|} \\
&= \frac{\lambda}{2} \int_0^1 \overline{h}_{X_\Phi(0)}(s) \overline{p}\,ds \\
&= \frac{\lambda}{2} \int_0^{+\infty} h(t) \overline{p} e^{-\lambda\overline{a}} \frac{(\lambda\overline{a})^{\lfloor t \rfloor}}{\lfloor t \rfloor !} dt
\end{aligned}$$

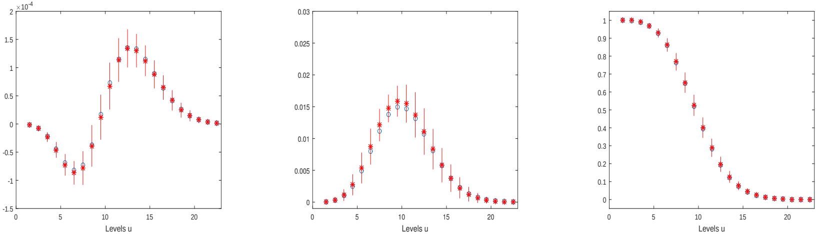
Hence for a.e. $t \geq 0$,

$$C_1^*(X,t) = e^{-\lambda\overline{a}} \frac{(\lambda\overline{a})^{\lfloor t \rfloor}}{\lfloor t \rfloor !} \frac{\lambda\overline{p}}{2}$$

Illustration

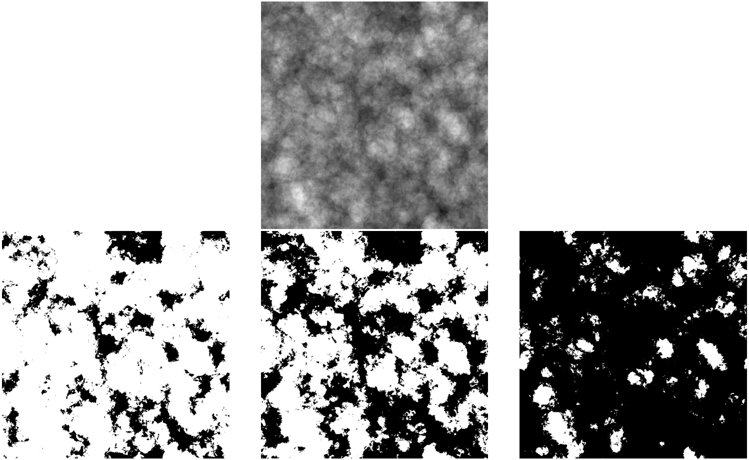
In a similar way, by continuity, for $t \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, it holds that

$$
\begin{aligned}
C_0^*(t) &= e^{-\lambda\bar{a}} \frac{(\lambda\bar{a})^{\lfloor t \rfloor}}{\lfloor t \rfloor !} \lambda \left(1 - \lambda \frac{\bar{p}^2}{4\pi} + \lfloor t \rfloor \frac{\bar{p}^2}{4\pi\bar{a}}\right), \\
C_1^*(t) &= e^{-\lambda\bar{a}} \frac{(\lambda\bar{a})^{\lfloor t \rfloor}}{\lfloor t \rfloor !} \frac{\lambda\bar{p}}{2} \\
C_2^*(t) &= e^{-\lambda\bar{a}} \sum_{k>t} \frac{(\lambda\bar{a})^k}{k!}
\end{aligned}
$$



Shot noise on a domain of size $2^{10} \times 2^{10}$ pixels, with intensity $\lambda = 0.0005$, and random disks of radius $R = 50$ or $R = 100$ (each with probability 0.5).

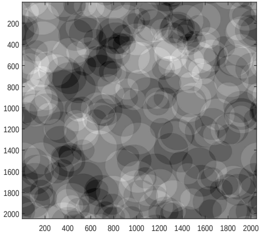
Illustration

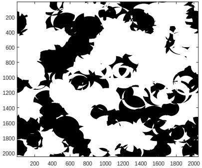


Critical levels

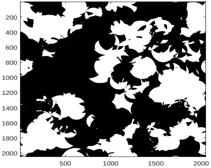
Extension

Taking $Y = X^{(1)} - X^{(2)}$ where $X^{(1)}, X^{(2)}$ are iid Shot noise of disks we have excursion sets that are not positive reach set but still elementary sets

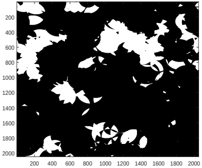




$t = -0.5$



$t = 0.5$

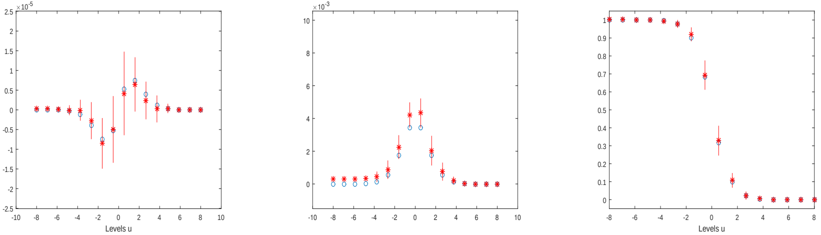


$t = 1.5$

LK densities

For $t \in \mathbb{R} \setminus \mathbb{Z}$, it holds that for $\nu = 2\lambda$ and $I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+n}}{m!\Gamma(m+n+1)}$ the modified Bessel function of the first kind

$$
\begin{aligned}
C_0^*(Y,t) &= \frac{\nu}{2}\, e^{-\nu\bar{a}} \, \Big[(I_{|\lfloor t \rfloor|} - I_{|\lfloor t \rfloor + 1|})(\nu\bar{a}) \\
&\qquad + \frac{\nu\bar{p}^2}{8\pi} \, (I_{|\lfloor t \rfloor - 1|} + I_{|\lfloor t \rfloor|} - I_{|\lfloor t \rfloor + 1|} - I_{|\lfloor t \rfloor + 2|})(\nu\bar{a}) \Big] \,, \\
C_1^*(Y,t) &= \frac{\nu\bar{p}}{4}\, e^{-\nu\bar{a}} \, (I_{|\lfloor t \rfloor|} + I_{|\lfloor t \rfloor + 1|})(\nu\bar{a}) \\
C_2^*(Y,t) &= e^{-\nu\bar{a}} \sum_{k>t} I_{|k|}(\nu\bar{a}),
\end{aligned}
$$



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Outlines

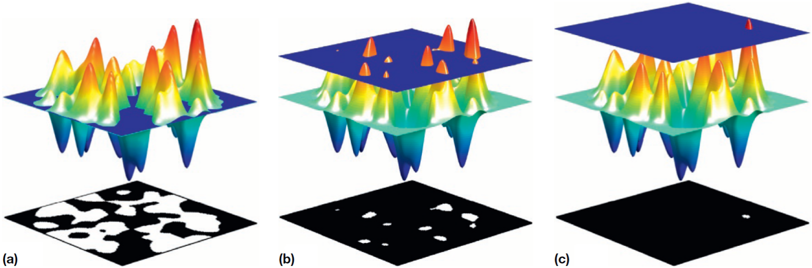
Lecture 2 :

Smooth Excursion sets

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2. For $t \in \mathbb{R}$, we consider the excursion set of level t

$$E_f(t) := \{x \in \mathbb{R}^2; f(x) \geq t\}.$$

We assume it is observed through U a bounded **open rectangle**.



Credit : BrainMapping : an encyclopedic reference- Topological Inference

Smooth excursion sets and implicit planar curves

Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2. Since f is continuous, we have for $t \in \mathbb{R}$

$$\partial E_f(t) = \{x \in \mathbb{R}^2 ; f(x) = t\},$$

corresponding to a level line of f. Hence for $t \in \mathbb{R}$, if $x \in \partial E_f(t)$ with $\nabla f(x) \neq 0$, the unit vector $\nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|}$ is the normal vector of $\partial E_f(t)$ at x with

$$D\nu_f(x) = \frac{1}{\|\nabla f(x)\|} \left[I_2 - \nu_f(x)\nu_f(x)^T \right] D^2 f(x),$$

where $D^2 f(x)$ is the Hessian matrix. Moreover the **signed curvature** at x may be written as

$$\kappa_f(x) = -\langle \nu_f(x)^\perp, D\nu_f(x)\nu_f(x)^\perp \rangle = -\frac{1}{\|\nabla f(x)\|} \langle \nu_f(x)^\perp, D^2 f(x)\nu_f(x)^\perp \rangle.$$

Coarea formula

By Morse-Sard theorem, the image by f of the set of critical values of f has measure 0 in $\mathbb{R}$. For a.e. level $t \in \mathbb{R}$ and U open bounded,

$$\begin{aligned}
\Phi_1(E_f(t), U) &= \frac{1}{2}\int_{\partial E_f(t)\cap U} 1\mathcal{H}^1(dx) \\
\Phi_0(E_f(t), U) &= \frac{1}{2\pi}\int_{\partial E_f(t)\cap U} \kappa_f(x)\mathcal{H}^1(dx).
\end{aligned}$$

The **coarea formula** states that, for any borel function $g : \mathbb{R}^2 \to \mathbb{R}$ s.t $\int_U |g(x)|\|\nabla f(x)\|\, dx < +\infty$,

$$\int_{\mathbb{R}}\int_{\partial E_f(t)\cap U} g(x)\mathcal{H}^1(dx)\, dt = \int_U g(x)\|\nabla f(x)\|\, dx.$$

Let us choose $h : \mathbb{R} \to \mathbb{R}$ a bounded continuous function (test function) such that multiplying $g(x)$ by $h(f(x))$ we get

$$\int_{\mathbb{R}} h(t)\int_{\partial E_f(t)\cap U} g(x)\mathcal{H}^1(dx)\, dt = \int_U h(f(x))g(x)\|\nabla f(x)\|\, dx.$$

Weak formula for Φ_1 and Φ_0

Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and $\int_U |g(x)| \|\nabla f(x)\| \, dx < +\infty$, recall the **Coarea formula** :

$$\int_{\mathbb{R}} h(t) \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^1(dx) \, dt = \int_U h(f(x)) g(x) \|\nabla f(x)\| \, dx.$$

Coarea formula with $g(x) = 1$:

$$\int_{\mathbb{R}} h(t) \Phi_1(E_f(t), U) dt = \frac{1}{2} \int_U h(f(x)) \|\nabla f(x)\| \, dx.$$

Coarea formula with $g(x) = \kappa_f(x) 1_{\|\nabla f(x)\| > 0}$ for

$$\kappa_f(x) = -\frac{1}{\|\nabla f(x)\|} \langle \nu_f(x)^\perp, D^2 f(x) \nu_f(x)^\perp \rangle, \text{ and } \nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|},$$

$$\int_{\mathbb{R}} h(t) \Phi_0(E_f(t), U) dt = -\frac{1}{2\pi} \int_U h(f(x)) \langle \nu_f(x)^\perp, D^2 f(x) \nu_f(x)^\perp \rangle 1_{\|\nabla f(x)\| > 0} \, dx.$$

Mean geometry for excursion sets

Let $X = (X(x))_{x\in\mathbb{R}^2}$ be a stationary real random field a.s. C^2 with $X(0)$, $\nabla X(0)$ and $D^2X(0)$ L^1 and $\mathbb{P}(\|\nabla X(0)\| = 0) = 0$. We consider the excursion set of level $t \in \mathbb{R}$

$$E_X(t) := \{x \in \mathbb{R}^2; X(x) \geq t\}.$$

First recall that by stationarity

$$\mathbb{E}[\Phi_2(E_X(t), U)] = \mathbb{E}\left(\int_U 1_{X(x)\geq t}dx\right) = |U|\mathbb{P}(X(0) \geq t).$$

Moreover, taking expectation it follows that for all h bounded continuous, writing $\nabla X(0) = \|\nabla X(0)\|\nu_X(0)$ a.s.,

$$\begin{aligned}
\int_{\mathbb{R}} h(t)\mathbb{E}[\Phi_1(E_X(t), U)]dt &= |U| \times \frac{1}{2}\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right) \\
\int_{\mathbb{R}} h(t)\mathbb{E}[\Phi_0(E_X(t), U)]dt &= |U| \times \frac{-1}{2\pi}\mathbb{E}\left(h(X(0))\langle\nu_X(0)^\perp, D^2X(0)\nu_X(0)^\perp\rangle\right)
\end{aligned}$$

Lipschitz-Killing (LK) densities

We therefore consider LK densities :

$$C_j^*(X,t) = \frac{1}{|U|}\mathbb{E}[\Phi_j(E_X(t),U)].$$

$$\begin{aligned}
\int_{\mathbb{R}} h(t)C_1^*(X,t)dt &= \frac{1}{2}\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right);\\
\int_{\mathbb{R}} h(t)C_0^*(X,t)dt &= \frac{-1}{2\pi}\mathbb{E}\left(h(X(0))\langle \nu_X(0)^{\perp}, D^2X(0)\nu_X(0)^{\perp}\rangle\right).
\end{aligned}$$

Note that $\int_{\mathbb{R}} C_1^*(X,t)dt = \frac{1}{2}\mathbb{E}\left(\|\nabla X(0)\|\right)$ (total variation)

Lipschitz-Killing (LK) densities

Let us remark that for any rotation Q,

$$C_j^*\left(\frac{X \circ Q^T - m}{\sigma}, t\right) = C_j^*(X, \sigma t + m).$$

such that for a second order random field X, we can assume that it is standard with $\Gamma_{\nabla X}$ diagonal matrix

- $\mathbb{E}(X(0)) = 0$
- $\mathrm{Var}(X(0)) = 1$
- $\mathrm{Cov}(X_i(0), X_j(0)) = \gamma_i \delta_{i,j}$

Here and in the sequel we write $X_i = \partial_i X$ and $X_{ij} = \partial_{ij} X$. Introducing Θ such that $\nu_X = (\cos(\Theta), \sin(\Theta))$ we also have

$$\mathbb{E}\left(h(X(0))\langle \nu_X(0)^\perp, D^2 X(0) \nu_X(0)^\perp \rangle\right)$$

$$= \mathbb{E}\left(h(X(0))\left[X_{11}(0)\sin^2(\Theta) + X_{22}(0)\cos^2(\Theta) - X_{12}(0)\sin(2\Theta)\right]\right)$$

LK densities at a fixed level

When $X(0)$ admits $p_{X(0)}$ for density one has for a.e. t

- $C_1^*(X,t) = \frac{1}{2}\mathbb{E}\left(\|\nabla X(0)\| | X(0)=t\right) p_{X(0)}(t)$
- $C_0^*(X,t) = -\frac{1}{2\pi}\mathbb{E}\left(\langle \nu_X(0)^\perp, D^2X(0)\nu_X(0)^\perp\rangle | X(0)=t\right) p_{X(0)}(t).$

Comments

- If one knows that $t \mapsto C_1^*(X,t)$ or $t \mapsto C_0^*(X,t)$ are continuous then a.e. is enough !
- In Berzin, Latour, Leon (2017) general assumptions to ensure that $t \mapsto C_1^*(X,t)$ is continuous ;
- For a fixed level $t \in \mathbb{R}$ one has to ensure that it is not a critical level

$$\mathbb{P}\left(\exists x \in \mathbb{R}^2 ; X(x) = t, \nabla X(x) = 0\right) = 0.$$

Bulinskaya's Lemma holds for instance when X is C^3 (see also D'Armenato, Azais, Leon, 2023 for weakest assumptions)

Outlines

LK densities for isotropic fields

When X is also **isotropic**, let introduce

$$J = \|\nabla X(0)\| e^{i\Theta} \text{ and } K = \frac{X_{22}(0) - X_{11}(0) + 2iX_{12}(0)}{4}.$$

The rotation invariance implies that for any $\theta \in [0, 2\pi)$,

$$(X(0), J, K) \stackrel{d}{=} (X(0), e^{i\theta} J, e^{2i\theta} K). \tag{1}$$

Writing $\|\nabla X(0)\| = \frac{1}{4} \int_0^{2\pi} |\Re(Je^{-i\theta})| d\theta$ we get

$$\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right) = \frac{\pi}{2} \mathbb{E}\left(h(X(0))|X_1(0)|\right).$$

Moreover,

$$\mathbb{E}\left(h(X(0))\langle \nu_X(0)^\perp, D^2 X(0) \nu_X(0)^\perp \rangle\right) = \alpha_0(h) + 2\Re\alpha_2(h),$$

with

$$\alpha_0(h) = \mathbb{E}\left(h(X(0))X_{11}(0)\right), \text{ and } \alpha_2(h) = \mathbb{E}\left(h(X(0))Ke^{2i\Theta}\right).$$

LK densities for isotropic fields

Noting that for all $k \neq 2$

$$\mathbb{E}\left(h(X(0))Ke^{ik\Theta}\right) = 0,$$

it follows that for any continuous bounded 2π periodic function

$$\mathbb{E}\left(h(X(0))Kg(\Theta)\right) = c_2(g)\mathbb{E}\left(h(X(0))Ke^{2i\Theta}\right),$$

with

$$c_2(g) = \frac{1}{2\pi}\int_0^{2\pi} e^{-2i\theta}g(\theta)d\theta.$$

In particular, for $g(\theta) = \sin(2\theta)$ we obtain

$$2\Re\alpha_2(h) = -2\mathbb{E}\left(h(X(0))X_{12}(0)\sin(2\Theta)\right).$$

Gaussian case

For X a stationary isotropic standard C^2 Gaussian random field we note $\rho(x) = \mathrm{Cov}(X(x), X(0))$, and the second spectral moment

$$\gamma_2 = -\partial_k^2 \rho(0) = -\mathrm{Cov}(X(0), X_{kk}(0)) = \mathrm{Var}(X_k(0)).$$

By stationarity $\mathrm{Cov}(X(0), X_k(0)) = \mathrm{Cov}(X_k(0), X_{ij}(0)) = 0$ and $\nabla X(0)$ is **independent** of $(X(0), D^2X(0))$

$$\mathbb{E}\left(h(X(0))|X_1(0)|\right) = \mathbb{E}\left(h(X(0))\right)\mathbb{E}\left(|X_1(0)|\right) = \sqrt{\frac{2\gamma_2}{\pi}}\mathbb{E}\left(h(X(0))\right);$$

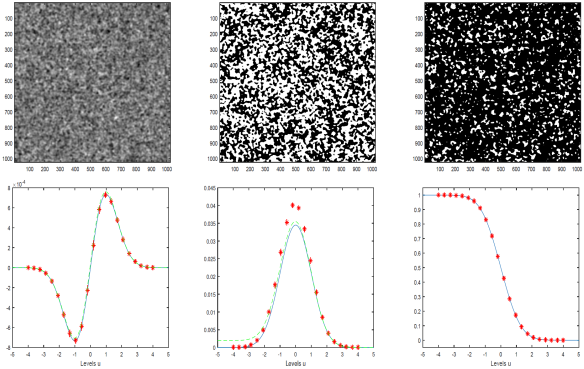
$$\begin{aligned} \alpha_0(h) &= \mathbb{E}\left(h(X(0))X_{11}(0)\right) = \mathbb{E}\left(h(X(0))\mathbb{E}\left(X_{11}(0)|X(0)\right)\right) \\ &= \frac{-\gamma_2}{\sigma^2}\mathbb{E}\left(h(X(0))X(0)\right); \end{aligned}$$

$$\begin{aligned} 2\Re\alpha_2(h) &= -\mathbb{E}\left(h(X(0))X_{12}(0)\sin(2\Theta)\right) = \mathbb{E}\left(h(X(0))X_{12}(0)\right)\mathbb{E}\left(\sin(2\Theta)\right) \\ &= 0. \end{aligned}$$

Gaussian case

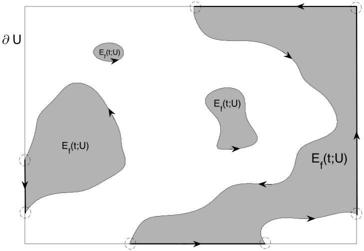
This yields to for a.e. $t \in \mathbb{R}$

$$C_0^*(X,t) = \frac{1}{(\sqrt{2\pi})^3}\, \gamma_2\, t\, e^{-\frac{t^2}{2}} \text{ and } C_1^*(X,t) = \frac{1}{4}\, \gamma_2^{1/2}\, e^{-\frac{t^2}{2}}.$$



$\rho(x) = e^{-\frac{\gamma_2}{2}\|x\|^2}$, for $\gamma_2 = 0.02$ in a domain of size $2^{10} \times 2^{10}$ pixels.

Statistical inference



In practise, it is often computed for $j = 0, 1, 2$,

$$C_j(X, t, \overline{U}) := \Phi_j(E_X(t) \cap \overline{U}, \overline{U}) \text{ (empirically accessible)}$$

Recall that

$$C_j(X, t, \overline{U}) = \Phi_j(E_X(t), U) + \Phi_j(E_X(t) \cap \overline{U}, \partial\overline{U}),$$

with $\mathbb{E}[\Phi_j(E_X(t), U)] = |U| \times C_j^*(X, t)$.

Kinematic formula

For **isotropic** random fields, $E_X(t)$ is an isotropic stationary closed random set

By kinematic formula (see Schneider, Weyl (2008)) under good assumptions, one has

$$\mathbb{E}[C_0(X,t,\overline{U})] = C_0^*(X,t)|\overline{U}| + \frac{1}{\pi} C_1^*(X,t)\mathcal{H}^1(\partial \overline{U}) + C_2^*(X,t),$$
$$\mathbb{E}[C_1(X,t,\overline{U})] = C_1^*(X,t)|\overline{U}| + \frac{1}{2} C_2^*(X,t)\mathcal{H}^1(\partial \overline{U}),$$
$$\mathbb{E}[C_2(X,t,\overline{U})] = C_2^*(X,t)|\overline{U}|$$

Hence unbiased estimators for $C_j^*(X,t)$ may be obtained as

$$\hat{C}_2(X,t) = \frac{1}{|\overline{U}|} C_2(X,t,\overline{U}),$$
$$\hat{C}_1(X,t) = \frac{1}{|\overline{U}|} C_1(X,t,\overline{U}) - \frac{\mathcal{H}^1(\partial \overline{U})}{2|\overline{U}|}\hat{C}_2(X,t),$$
$$\hat{C}_0(X,t) = \frac{1}{|\overline{U}|} C_0(X,t,\overline{U}) - \frac{\mathcal{H}^1(\partial \overline{U})}{\pi|\overline{U}|}\hat{C}_1(X,t) - \frac{1}{|\overline{U}|}\hat{C}_2(X,t).$$

Central limit theorems

Under good assumptions one can get

$$\frac{1}{\sqrt{|\overline{U}|}}\left(C_j(X,t,\overline{U}) - \mathbb{E}(C_j(X,t,\overline{U})\right) \underset{\overline{U}\nearrow\mathbb{R}^2}{\longrightarrow} \mathcal{N}(0,\sigma_j^2(t)),$$

leading to

$$\sqrt{|\overline{U}|}\left(\hat{C}_j(X,t) - C_j^*(X,t)\right) \underset{\overline{U}\nearrow\mathbb{R}^2}{\longrightarrow} \mathcal{N}(0,\sigma_j^2(t)).$$

Some Ref on CLT : Spodarev (2012), Estrade, León (2016), Müller (2017), Kratz Vadlamani (2018), Reddy et al (2018), Berzin (2021)...

Gaussian Kinematic formula

For Gaussian type random fields, $X = F(G)$ with $F : \mathbb{R}^\ell \to \mathbb{R}$ and $G = (G_1, \dots, G_\ell)$ with $G_1, \dots, G_\ell$ iid smooth standard Gaussian field with $\mathrm{Var}(\nabla G_i) = \Gamma_\nabla$ the Gaussian kinematic formula (see Adler, Taylor (2007)) states under good assumptions,

$$
\begin{aligned}
\mathbb{E}[\mathcal{L}_0^\nabla(X, t, \overline{U})] &= \frac{1}{2\pi}\mathcal{M}_2(X,t)\mathcal{L}_2^\nabla(\overline{U}) + \frac{1}{\sqrt{2\pi}}\mathcal{M}_1(X,t)\mathcal{L}_1^\nabla(\overline{U}) \\
&\quad + \mathcal{M}_0(X,t)\mathcal{L}_0^\nabla(\overline{U}), \\
\mathbb{E}[\mathcal{L}_1^\nabla(X, t, \overline{U})] &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\mathcal{M}_1(X,t)\mathcal{L}_2^\nabla(\overline{U}) + \mathcal{M}_0(X,t)\mathcal{L}_1^\nabla(\overline{U}), \\
\mathbb{E}[\mathcal{L}_2^\nabla(X, t, \overline{U})] &= \mathcal{M}_0(X,t)\mathcal{L}_2^\nabla(\overline{U})
\end{aligned}
$$

with

$$
\mathbb{P}\left(G(0) \in Tube(F,\rho)\right) = \mathcal{M}_0(X,t) + \rho\mathcal{M}_1(X,t) + \frac{1}{2}\rho^2\mathcal{M}_2(X,t) + O(\rho^3),
$$

and

$$
Tube(F,\rho) := \{g \in \mathbb{R}^\ell ; \mathrm{dist}\left(g, F^{-1}([t,+\infty))\right) \le \rho\}.
$$

(Gaussian) Kinematic formula

For isotropic fields $\Gamma_\nabla = \gamma_2 I_2$ and

$$\mathbb{E}[\mathcal{L}_j^\nabla(X, t, \overline{U})] = \gamma_2^{j/2} C_j(X, t, \overline{U}).$$

It follows that

$$\begin{aligned} C_0^*(X, t) &= \frac{\gamma_2}{2\pi}\mathcal{M}_2(X, t) \\ C_1^*(X, t) &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\gamma_2}\mathcal{M}_1(X, t) \\ C_2^*(X, t) &= \mathcal{M}_0(X, t) \end{aligned}$$

Remark that for X standard Gaussian, $k = 1$, $G = X$ and $F = Id$, one has

$$\mathbb{P}\left(G(0) \in Tube(F, \rho)\right) = 1 - \Psi(t - \rho),$$

$$\mathcal{M}_0(X, t) = 1 - \Psi(t), \mathcal{M}_1(X, t) = \Psi'(t), \mathcal{M}_2(X, t) = -\Psi''(t)$$

Chi2 case

For $k \geq 1$, $Z_k = G_1^2 + \ldots + G_k^2$ and normalized field

$$\widetilde{Z}_k(x) := \frac{1}{\sqrt{2k}}(Z_k(x) - k),\ x \in \mathbb{R}^2.$$

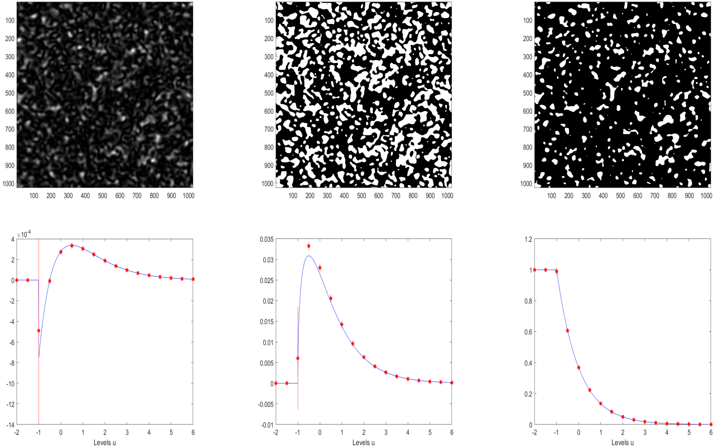
Then, for all $t > -\sqrt{k/2}$, $C_j^*(\widetilde{Z}_k, t) = C_j^*(Z_k, k + t\sqrt{2k})$ and

$$C_0^*(Z_k, t) = \frac{\gamma_2}{\pi 2^{k/2}\Gamma(k/2)} t^{(k-2)/2}(t + 1 - k)\exp\left(-\frac{t}{2}\right),$$

$$C_1^*(Z_k, t) = \frac{\sqrt{\pi\gamma_2}}{2^{(k+1)/2}\Gamma(k/2)} t^{(k-1)/2}\exp\left(-\frac{t}{2}\right),$$

$$C_2^*(Z_k, t) = \int_t^{+\infty} \frac{1}{2^{k/2}\Gamma(k/2)} u^{(k-2)/2} exp\left(-\frac{u}{2}\right) du,$$

Chi2 case



$\widetilde{Z}_k$ for $k = 2$ and for iid standard $G_1, \dots, G_k$ with covariance function $\rho(x) = e^{-\frac{\gamma_2}{2}\|x\|^2}$, for $\gamma_2 = 0.02$ in a domain of size $2^{10} \times 2^{10}$ pixels.

Student case

For $k \geq 3$, $T_k = G_{k+1}/\sqrt{Z_k/k}$ and normalized field

$$\widetilde{T}_k(x) := \sqrt{(k-2)/k}\, T_k(x),\ x \in \mathbb{R}^2.$$

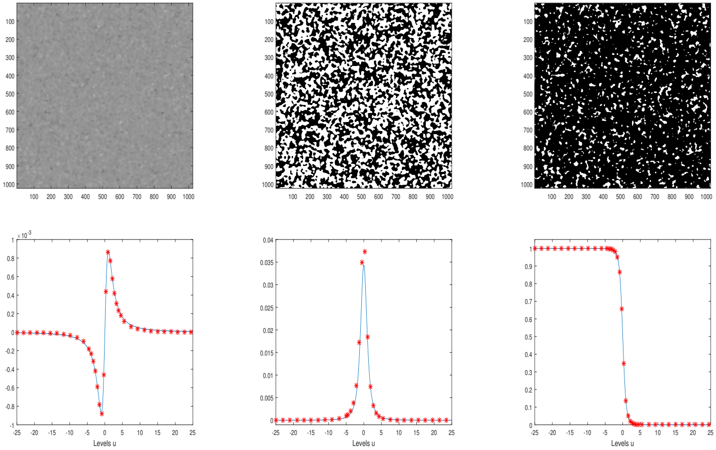
Then, $C_j^*(\widetilde{\mathcal{T}}_k, t) = C_j^*(\mathcal{T}_k, t\sqrt{k/(k-2)})$

$$C_0^*(\widetilde{\mathcal{T}}_k, t) = \frac{\gamma_2(k-1)}{4\pi} \frac{t}{\sqrt{k\pi}} \frac{\Gamma\left(\frac{k-1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{t^2}{k}\right)^{-\frac{k-1}{2}},$$

$$C_1^*(\widetilde{\mathcal{T}}_k, t) = \frac{\sqrt{\gamma_2}}{4} \left(1 + \frac{t^2}{k}\right)^{-\frac{k-1}{2}},$$

$$C_2^*(\widetilde{\mathcal{T}}_k, t) = \int_t^{+\infty} \frac{1}{\sqrt{k\pi}} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{u^2}{k}\right)^{-\frac{k+1}{2}} du.$$

Student case



$\widetilde{T}_k$ for $k = 4$ and and for iid standard $G_1, \dots, G_{k+1}$ with covariance function $\rho(x) = e^{-\frac{\gamma_2}{2}\|x\|^2}$, for $\gamma_2 = 0.02$ in a domain of size $2^{10} \times 2^{10}$ pixels.

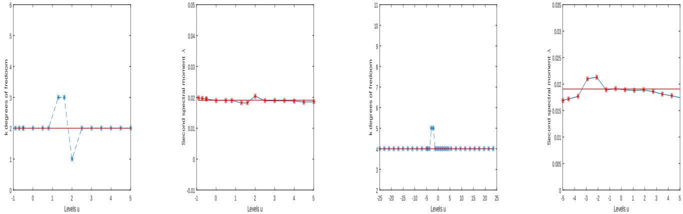
Spectral moment estimate

Using the fact that $\gamma_2 = \frac{2\pi}{\mathcal{M}_{\mathbf{2}}(X,t)} C_0^*(X,t)$ we can estimate it from a single excursion estimating first k using

$$\hat{k} = \mathrm{argmin}_k |\hat{C}_2(X,t) - C_2^*(X,k,t)|$$

and then plugging

$$\hat{\gamma}_2 = \frac{2\pi}{\mathcal{M}_2(X,\hat{k},t)} \hat{C}_0(X,t)$$



First line for $\chi^2(k)$ with $k = 2$ and second line for $\mathrm{Stu}(k)$ with $k = 4$ and $\gamma_2 = 0.02$

Shot noise random fields

Let X be a smooth **(Poisson) shot noise random field** given by

$$\forall x \in \mathbb{R}^d, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i), \text{ where}$$

- $\{x_i\}_{i \in I}$ is a Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^2$,
- $\{m_i\}_{i \in I}$ are independent « marks » with distribution $F(dm)$ on $\mathbb{R}^\ell$, and independent of $\{x_i\}_{i \in I}$.
- The functions g_m are C^3 with for $|\mathrm{j}| \leq 3$

$$\int_{\mathbb{R}^\ell} \int_{\mathbb{R}^2} |\partial_{\mathrm{j}}^{|\mathrm{j}|} g_m(y)| dy F(dm) < +\infty$$

Shot noise fields

Recall that the characteristic function of $X(x)$ is given by

$$\mathbb{E}\left(e^{i\xi X(x)}\right) = \exp\left(\lambda \int_{\mathbb{R}^\ell \times \mathbb{R}^2} [e^{i[\xi g_m(y)]} - 1] F(dm) dy\right).$$

When g is smooth, we have also access to joint law of $(X(x), \nabla X(x), D^2 X(x))$ via characteristic function and similar integral expression. In particular the joint characteristic function of $X(x)$ and $\partial_1 X(x)$ is

$$\begin{aligned} \varphi(\xi, s) &= \mathbb{E}\left(e^{i\xi X(x) + is\partial_1 X(x)}\right) \\ &= \exp\left(\lambda \iint [e^{i\xi g_m(y) + is\partial_1 g_m(y)} - 1]\, F(dm)\, dy\right) \end{aligned}$$

Isotropic smooth Shot noise fields

The main idea is therefore to take $h_\xi(t) = e^{it\xi}$ to compute $\mathcal{F}C_j^*(\xi) = \int_{\mathbb{R}} e^{it\xi} C_j^*(X,t)dt$. We obtain integral formulas :

$$\mathcal{F}C_1^*(\xi) = \frac{1}{2}\int_0^{+\infty} \frac{\varphi(\xi,s)}{s} S_1(\xi,s)ds.$$

$$\mathcal{F}C_0^*(\xi) = S_0(\xi)\varphi(\xi,0) + \int_0^{+\infty} \frac{\varphi(\xi,s)}{s} S_2(\xi,s)ds,$$

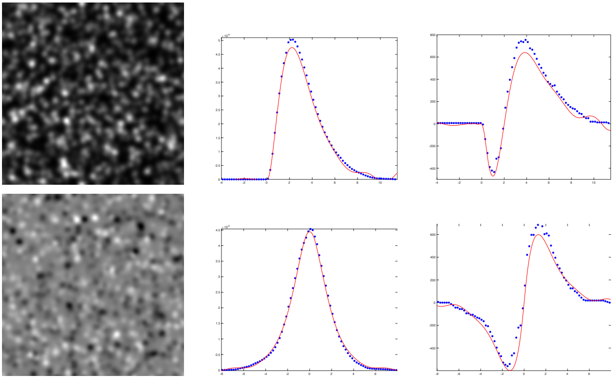
with

$$\begin{aligned}
S_1(\xi) &= -i\lambda \int_{\mathbb{R}^\ell}\int_{\mathbb{R}^2} \partial_1 g_m(y) e^{i[\xi g_m(y) + s\partial_1 g_m(y)]}\, dy\, F(dm) \\
S_0(\xi) &= -\frac{\lambda}{2\pi}\int_{\mathbb{R}^\ell}\int_{\mathbb{R}^2} \partial_1^2 g_m(y) e^{i\xi g_m(y)}\, dy\, F(dm) \\
S_2(\xi,s) &= \frac{\lambda}{2\pi}\int_{\mathbb{R}^\ell}\int_{\mathbb{R}^2} [\partial_2^2 g_m(y) - \partial_1^2 g_m(y)] e^{i[\xi g_m(y) + s\partial_1 g_m(y)]}\, dy\, F(dm)
\end{aligned}$$

Shot noise Gaussian examples

We choose $g_m(y) = me^{-\frac{\|y\|^2}{2\sigma^2}}$ for $m \in \mathbb{R}$ and with $F(dm)$ distribution of M

- Top : $M \sim \mathcal{E}(\mu)$, we find $\varphi(t) = \left(\frac{\mu}{\mu - it}\right)^{2\pi\lambda\sigma^2}$ and $X(x) \sim \gamma(\mu, 2\pi\lambda\sigma^2)$;
- Bottom : $M \sim \mathcal{L}(\mu)$, $\varphi(t) = \left(\frac{\mu^2}{\mu^2 + t^2}\right)^{\pi\lambda\sigma^2}$ and $X(x) \sim GS\mathcal{L}(\mu, \pi\lambda\sigma^2)$.



Outlines

Anisotropic smooth Gaussian fields

Let $X = (X(x))_{x \in \mathbb{R}^2}$ be a C^2 stationary Gaussian random field. We denote as usual $E_X(t)$ the excursion set of level $t \in \mathbb{R}$
Consider for instance the covariance function

$$\rho(x) = e^{-\frac{\gamma_1}{2} x_1^2} e^{-\frac{\gamma_2}{2} x_2^2}.$$

$\gamma_1 = \gamma_2 = 0.005$ $\quad t = 0$ $\quad t = 1$ $\quad t = 1.96$

$\gamma_1 = 0.002$

Lipschitz-Killing (LK) densities

When $X(0)$ admits a density, recall our general formulas for a.e. t

- $C_1^*(X,t) = \frac{1}{2}\mathbb{E}\left(\|\nabla X(0)\| | X(0) = t\right) p_{X(0)}(t)$
- $C_0^*(X,t) = -\frac{1}{2\pi}\mathbb{E}\left(\langle \nu_X(0)^\perp, D^2X(0)\nu_X(0)^\perp\rangle | X(0) = t\right) p_{X(0)}(t)$.

Assume that X is standard and $\Gamma_{\nabla X} = \mathrm{diag}(\gamma_1, \gamma_2)$ is diagonal
Then $X_1(0)$ and $X_2(0)$ are independent and $X_{12}(0)$ is also independent from $X(0)$, and recall that $\nabla X(0)$ is **independent** from $X(0)$ and $D^2X(0)$ with

$$\mathbb{E}\left(X_{ii}(0) | X(0) = t\right) = -\gamma_i t.$$

Geometrical spectral moment

Writing $e_\theta = (\cos(\theta), \sin(\theta))$, we use

$$\mathbb{E}(\|\nabla X(0)\|) = \frac{1}{4}\int_0^{2\pi} \mathbb{E}(|\langle \nabla X(0), e_\theta\rangle|)d\theta,$$

with

$$\langle \nabla X(0), e_\theta\rangle = X_1(0)\cos(\theta) + X_2(0)\sin(\theta),$$

$\langle \nabla X(0), e_\theta\rangle \sim \sqrt{\gamma_1 \cos^2(\theta) + \gamma_2 \sin^2(\theta)}\mathcal{N}(0,1)$ with $\mathbb{E}(|\mathcal{N}(0,1)|) = \sqrt{\frac{2}{\pi}}$.

Proposition

$C_1^*(X,t) = \frac{1}{4}\sqrt{\gamma_{\mathrm{Per}}}e^{-t^2/2}$, *a.e.* $t \in \mathbb{R}$, *where*

$$\gamma_{\mathrm{Per}} = \left(\frac{1}{2\pi}\int_0^{2\pi} \sqrt{\gamma_1 \cos^2(\theta) + \gamma_2 \sin^2(\theta)}d\theta\right)^2.$$

Geometrical spectral moment

Let $\nu_X(0) = (\cos(\Theta), \sin(\Theta))$ with Θ independent from $X(0)$, $D^2X(0)$,

$$\begin{aligned} & \mathbb{E}\left(\left[X_{11}(0)\sin^2(\Theta) + X_{22}(0)\cos^2(\Theta) - X_{12}(0)\sin(2\Theta)\right] | X(0) = t\right) \\ = \quad & \left[-\gamma_1 t\mathbb{E}\left(\sin^2(\Theta)\right) - \gamma_2 t\mathbb{E}\left(\cos^2(\Theta)\right)\right] \end{aligned}$$

Proposition

$C_0^*(X,t) = \frac{1}{(2\pi)^{3/2}}\gamma_{\mathrm{TC}}te^{-t^2/2}$, *a.e.* $t \in \mathbb{R}$, *where*

$$\gamma_{\mathrm{TC}} = \mathbb{E}(\gamma_1 \sin^2(\Theta) + \gamma_2 \cos^2(\Theta)) = \sqrt{\gamma_1\gamma_2}.$$

Rk : if $\gamma_1 = \gamma_2$ then $\gamma_{\mathrm{TC}} = \gamma_{\mathrm{Per}} = \gamma_2$ and $\nu_X(0) \sim \mathcal{U}(S^1)$.

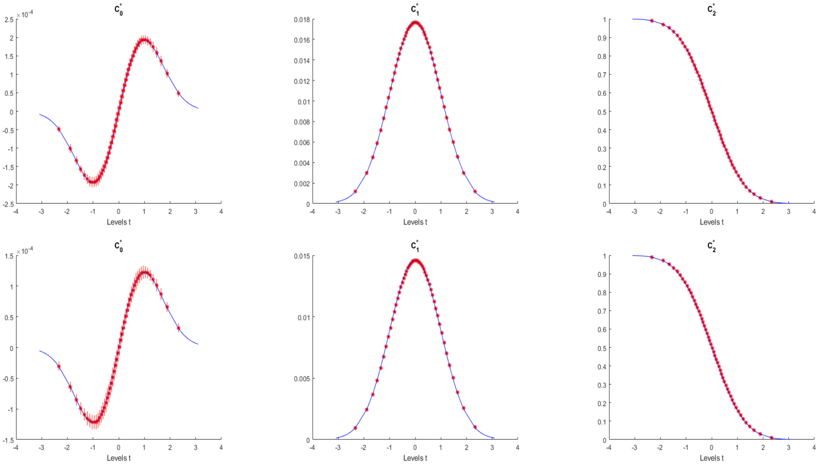
Gaussian Lipschitz-Killing (LK) densities

Theorem

For X C^2 stationary Gaussian standard random field

$$
\begin{aligned}
C_0^*(X,t) &= \gamma_{\mathrm{TC}} \frac{1}{(2\pi)^{3/2}}\; t\, e^{-\frac{t^2}{2}} \;\; a.e. \\
C_1^*(X,t) &= \sqrt{\gamma_{\mathrm{Per}}}\frac{1}{4}\; e^{-\frac{t^2}{2}} \;\; a.e. \\
C_2^*(X,t) &= 1-\Psi(t) \;\; \textit{for } \Psi(t) = \int_{-\infty}^{t} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du
\end{aligned}
$$

How to see anisotropy ?



C_0^* C_1^* C_2^*

First line : $\gamma_1 = \gamma_2 = 0.005$ and Second line $\gamma_1 = 0.001$

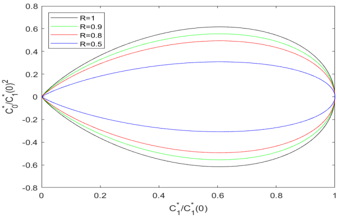
Ratio of anisotropy

Proposition

$\min(\gamma_1, \gamma_2) \le \gamma_{\mathrm{TC}} \le \gamma_{\mathrm{Per}} \le \max(\gamma_1, \gamma_2)$ and $\gamma_{\mathrm{TC}} = \gamma_{\mathrm{Per}}$ iff $\gamma_1 = \gamma_2$.

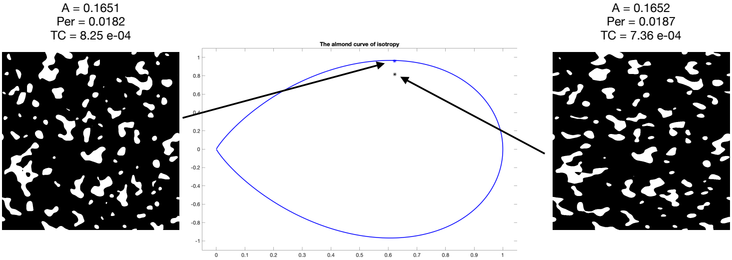
Defining $\mathrm{R} = \frac{\gamma_{\mathrm{TC}}}{\gamma_{\mathrm{Per}}} \in \left[\frac{\min(\gamma_1,\gamma_2)}{\max(\gamma_1,\gamma_2)}; 1\right]$ and plot the **Almond curve of anisotropy** $\{(x(t), y(t)); t \in \mathbb{R}\}$

$$x(t) = \frac{C_1^*(X,t)}{C_1^*(X,0)} = e^{-t^2/2} \text{ and } y(t) = \frac{C_0^*(X,t)}{(C_1^*(X,0))^2} = \frac{16}{(2\pi)^{3/2}} \mathrm{R}\, te^{-t^2/2}.$$

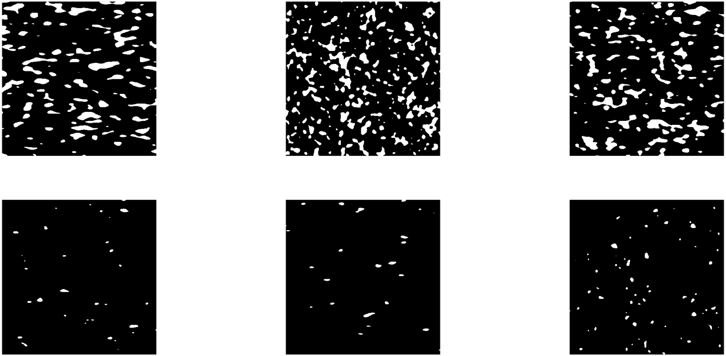


with $C_1^*(X,0) = 4\sqrt{\gamma_{\mathrm{Per}}}$. See also Klatt, Hörmann, Mecke (2021) for inspiration

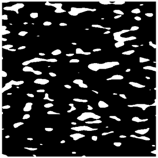
Is it anisotropic ?



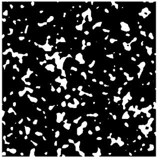
Is it anisotropic ?



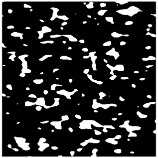
Is it anisotropic ?



R = 0.8

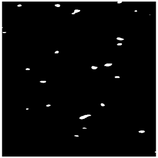


R = 1

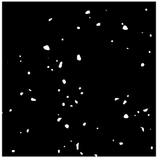


R = 0.9

R = 0.9



R = 0.8



R = 1

Effective level

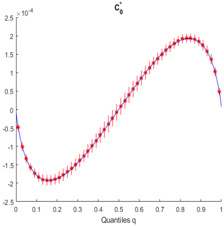
For $t \in \mathbb{R}$ unknown, following Di Bernardino and Duval (2020), define the effective level as

$$\hat{t} = \Psi^{-1}(1 - \hat{C}_2(X, t)),$$

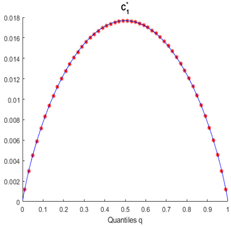
Note that for the quantile $t = \Psi^{-1}(q)$ for $q \in (0, 1)$ one has $C_2^*(X, t) = 1 - q$ and set

$$\hat{q} = 1 - \hat{C}_2(X, t) \text{ such that } \hat{t} = \Psi^{-1}(\hat{q}).$$

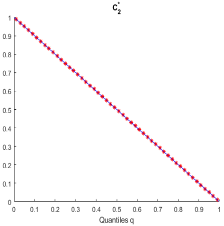
We can consider $C_j^*(X, \Psi^{-1}(q))$, $j = 0, 1, 2$.



$C_0^* \circ \Psi^{-1}$



$C_1^* \circ \Psi^{-1}$



$C_2^* \circ \Psi^{-1}$

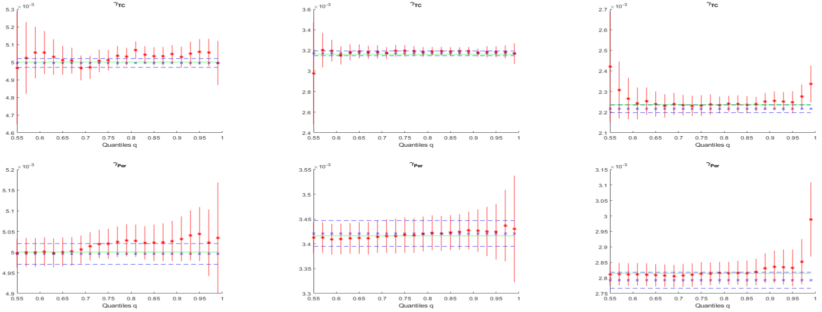
Effective γ_{Per} and γ_{TC}

Using that

$$C_0^*(X,t) = \gamma_{\mathrm{TC}} \frac{1}{(2\pi)^{3/2}}\ t\, e^{-\frac{t^2}{2}} \text{ and } C_1^*(X,t) = \sqrt{\gamma_{\mathrm{Per}}}\frac{1}{4}\ e^{-\frac{t^2}{2}},$$

define for $\hat{t} > 0$ or $\hat{q} > 1/2$

$$\hat{\gamma}_{\mathrm{TC}} = \hat{C}_0(X,t) \times (2\pi)^{3/2}\ \hat{t}^{-1}\, e^{\frac{\hat{t}^2}{2}} \text{ and } \hat{\gamma}_{\mathrm{Per}} = \hat{C}_1(X,t)^2 \times 16\ e^{\hat{t}^2}.$$

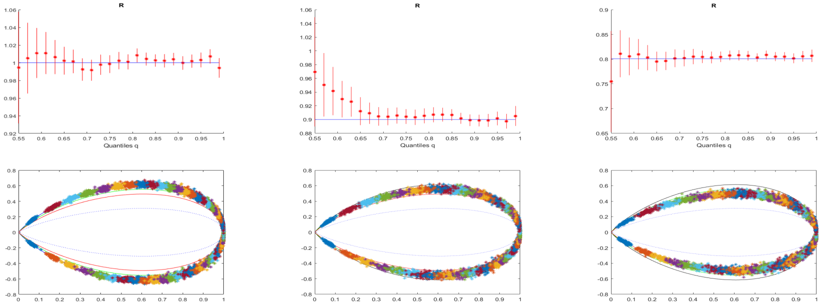


R = 1 R = 0.9 R = 0.8

Effective Ratio of anisotropy

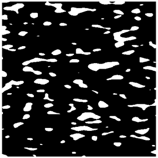
We finally define

$$\hat{\mathrm{R}} = \frac{\hat{\gamma}_{\mathrm{TC}}}{\hat{\gamma}_{\mathrm{Per}}} = \frac{\hat{C}_0(X,t)}{\hat{C}_1(X,t)^2} \times \frac{(2\pi)^{3/2}}{16} \; \hat{t}^{-1} \, e^{-\frac{\hat{t}^2}{2}}$$

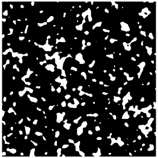


$\mathrm{R} = 1$ $\quad$ $\mathrm{R} = 0.9$ $\quad$ $\mathrm{R} = 0.8$

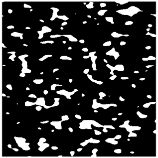
Is it anisotropic ?



$\hat{\mathrm{R}} = 0.7672$

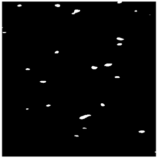


$\hat{\mathrm{R}} = 1.012$

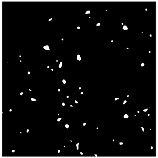


$\hat{\mathrm{R}} = 0.8826$

$\hat{\mathrm{R}} = 0.9029$



$\hat{\mathrm{R}} = 0.7372$



$\hat{\mathrm{R}} = 0.9762$

Conclusion and perspectives

Conclusion :

- New geometrical equivalent of spectral moments
- Anisotropy estimation available from one excursion set
- Extension in dimension d with mean curvature, numerical evaluation for $d = 3$

Perspectives :

- Second order and higher moment properties
- Control of bias induced by discrete simulation/estimation
- Extension for fractional Gaussian fields

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