The Geometry of Time Dependent Spherical Random Fields

Fall School "Time Series, Random Fields and beyond"

Ulm University, 23-27 September 2024

Anna Vidotto

Dipartimento di Matematica e Applicazioni
Università di Napoli "Federico II"

based on joint works with D. Marinucci and M. Rossi

The starting point

- Sphere-cross-time random fields

$$\left\{ Z_t(x),\ x \in \mathbb{S}^2 \right\}_t \qquad \mathbb{S}^2 :=$$

- Applications in **Geophysics, Climate Science** $\longrightarrow$ $\mathbb{S}^2$ represents the Earth
  - see the book *Random Field Models in Earth Sciences*, Christakos (2005)
  - Examples: Earth temperature, rainfall, CO_2 concentration

Outline

1. Possible questions
    Geometry of the excursion sets
    Statistical modelling
2. Answering the geometric questions
    Conditions on the field
    Chaos expansions and results
3. A look at some statistical questions
4. Future perspectives

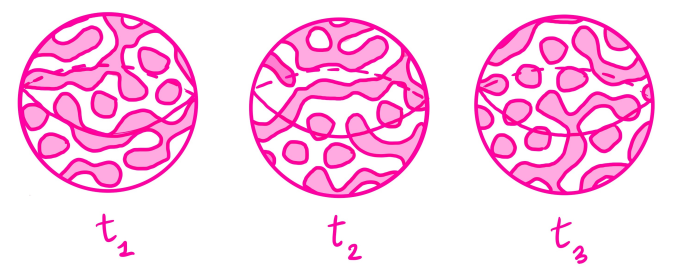
Possible questions

Geometry of the excursion sets

- $Z = \left\{ Z_t(x) : x \in \mathbb{S}^2, t \in \mathbb{R} \right\}$ Gaussian random field on $(\Omega, \mathcal{F}, \mathbb{P})$
- fix t and u, we want to study

$$\mathcal{E}_u(t) := \left\{ x \in \mathbb{S}^2 : Z_t(x) \geq u \right\}$$

$$\mathcal{B}_u(t) := \left\{ x \in \mathbb{S}^2 : Z_t(x) = u \right\}$$



Area and boundary length

- as t varies, $\mathcal{E}_u(t)$ and $\mathcal{B}_u(t)$ are **geometric random processes**
- fix $u \in \mathbb{R}$

$$\mathcal{A}_u(t) := \text{area}\,(\mathcal{E}_u(t)) = \text{area}\,(\{Z(\cdot,t) \geq u\}) \qquad \mathcal{L}_u(t) := \text{length}\,(\mathcal{B}_u(t)) = \text{length}\,(\{Z(\cdot,t) = u\})$$

$$\mathcal{M}_T(u) := \int_0^T \Big(\mathcal{A}_u(t) - \mathbb{E}[\mathcal{A}_u(t)]\Big)\,dt$$

$$\mathcal{C}_T(u) := \int_0^T \Big(\mathcal{L}_u(t) - \mathbb{E}[\mathcal{L}_u(t)]\Big)\,dt$$

Outline of the geometrical questions

- Goal: fluctuations of $\mathcal{A}_u(t)$ and $\mathcal{L}_u(t)$ around their mean
  - variance of $\mathcal{M}_T(u), \mathcal{C}_T(u)$ as $T \to \infty$
  - distribution of $\mathcal{M}_T(u), \mathcal{C}_T(u)$ as $T \to \infty$
- Main technique: Wiener chaos expansion
  - asymptotic behavior of $\mathcal{M}_T(u)$ and $\mathcal{C}_T(u)$ as $T \to \infty$ governed by an interplay between the level u and some memory parameters

Statistical modelling

- Looking at $Z_t(x) = Z_t(x)$ as a functional time series

$$\{Z_t(\cdot)\}_{t\in\mathbb{Z}} \qquad \text{discrete time}$$

- Possible goals:
  - Model estimation: OLS, LASSO...
  - Statistical testing of properties of the time series: non-stationarity, structural breaks...

Answering the geometric questions

Conditions on the field

- $\mathbb{E}[Z_t(x)] = 0, \quad \forall\, x \in \mathbb{S}^2, t \in \mathbb{R}$
- $\mathbb{E}\left[Z_t(x)Z_s(y)\right] = \Gamma\left(\langle x, y\rangle, t - s\right), \quad \forall\, x, y \in \mathbb{S}^2, t, s \in \mathbb{R}$
  - $\langle x, y\rangle = \cos d(x, y),\ d(x, y)$:=angle between x and y
  - $\Gamma : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ positive semidef fct

  $\iff$ Z isotropic in space and stationary in time
- Γ continuous $\iff$ Z mean-square continuous

Multipole expansion of the field

$$Z_t(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \underbrace{a_{\ell m}(t)}_{\text{random}} \underbrace{Y_{\ell m}(x)}_{\text{deterministic}} = \sum_{\ell=0}^{\infty} Z_{\ell;t}(x)$$

- spherical harmonics $\{Y_{\ell m} : \ell \geq 0, m = -\ell, \dots, \ell\}$ o.b. on $L^2(\mathbb{S}^2)$
- $a_{\ell m}(t) := \int_{\mathbb{S}^2} Z_t(x) Y_{\ell m}(x)\, dx$
- $\{a_{\ell m} : \ell \geq 0, m = -\ell, \dots, \ell\}$ stationary Gaussian processes:
  - $a_{\ell m} \perp\!\!\!\perp a_{\ell' m'}$ if either $\ell \neq \ell'$ or $m \neq m'$
  - $\mathbb{E}[a_{\ell m}(t) a_{\ell m}(s)] = C_\ell(t-s)$

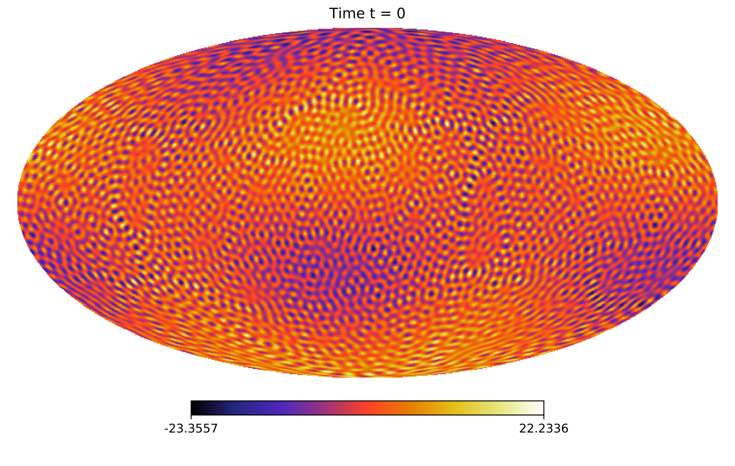
Representation of the covariance function

$$\Gamma\left(\langle x, y\rangle, t-s\right) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell(t-s) P_\ell(\langle x, y\rangle) = \sum_{\ell=0}^{\infty} \Gamma_\ell$$

- P_ℓ is the ℓ-th Legendre polynomial
- Γ_ℓ is the covariance function of the random field Z_ℓ
- ℓ represents the order of the so-called *multipole* $\rightsquigarrow$ spatial scale π/ℓ

Simulations

role of ℓ vs role of τ: positive correlation in time ($C_\ell(\tau)>0$) at large scales (small ℓ) and negative correlation in time ($C_\ell(\tau)<0$) at small scales (large ℓ)



Images by Alessia Caponera

The memory of Z_ℓ

$$C_\ell(\tau) \sim C_\ell(0)\, g_{\beta_\ell}(\tau) \qquad \tau \to +\infty$$

$$g_{\beta_\ell}(\tau) = \frac{1}{(1+|\tau|)^{\beta_\ell}}$$

- **long memory**: $\beta_\ell \in (0,1]$ $\rightsquigarrow$ non-integrable covariance
- **short memory**: $\beta_\ell \in (1,+\infty)$ $\rightsquigarrow$ integrable covariance

The memory parameter

- β_ℓ is the memory parameter
- **assumption:** $\exists \beta_{\ell^\star} := \min\{\beta_\ell, \ell \in \mathbb{N}, \ell \geq 1\}$ (excluding β_0)
- $\beta_{\ell^\star}$ = smallest exponent = largest memory
- $\mathcal{I}^\star := \{\ell \in \mathbb{N} : \beta_\ell = \beta_{\ell^\star}\}$ ($\ell = 0$ can belong to $\mathcal{I}^\star$)
- $\mathcal{I}^\star$ are the multipoles that mostly contribute to the covariance function

The Wiener chaos built from the $a_{\ell m}(t)$'s

- $\mathbb{A} := \overline{\text{lin}\,\{a_{\ell m}(t), \ell \geq 0, |m| \leq \ell, t \in [0,T]\}}^{\|\cdot\|_{L^2(\mathbb{P})}}$

- $q \in \mathbb{N}_0$, q-th Wiener chaos: $\mathcal{H}_q := \overline{\text{lin}\,\{H_q(\xi), \xi \in \mathbb{A} \sim \mathcal{N}(0,1)\}}^{\|\cdot\|_{L^2(\mathbb{P})}}$

$$\mathcal{H}_q \perp \mathcal{H}_{q'} \qquad q \neq q' \qquad\qquad F \in L^2(\Omega, \sigma(\mathbb{A}), \mathbb{P})$$

$$L^2(\Omega, \sigma(\mathbb{A}), \mathbb{P}) = \bigoplus_{q=0}^{+\infty} \mathcal{H}_q \quad \Longrightarrow \quad F = \sum_{q=0}^{+\infty} F[q] =: \sum_{q=0}^{+\infty} \text{proj}\,[F|\mathcal{H}_q]$$

- $\mathcal{H}_0 = \mathbb{R}$, $F[0] = \mathbb{E}[F]$, $\text{Var}\,F = \sum_{q=0}^{+\infty} \text{Var}(F[q])$

Chaotic decomposition of $\mathcal{M}_T(u)$

- **Recall**: $\mathcal{M}_T(u) = \int_0^T (\mathcal{A}_u(t) - \mathbb{E}[\mathcal{A}_u(t)])dt$
- $\mathcal{M}_T(u) \in L^2(\mathbb{P}) \implies \mathcal{M}_T(u) = \sum_{q=0}^{+\infty} \mathcal{M}_T(u)[q]$

$$\mathcal{M}_T(u)[q] = \frac{H_{q-1}(u)\phi(u)}{q!} \int_0^T \int_{\mathbb{S}^2} H_q\left(Z_t(x)\right) dx dt$$

- $\mathcal{M}_T(0)[2] \equiv 0$

Chaotic decomposition of $\mathcal{C}_T(u)$

- **Recall**: $\mathcal{C}_T(u) = \int_0^T (\mathcal{L}_u(t) - \mathbb{E}[\mathcal{L}_u(t)])dt$

$$\mathcal{C}_T(u) = \sigma_1 \sum_{q=1}^{+\infty} \sum_{m=0}^{q} \sum_{k=0}^{m} \frac{\alpha_{k,m-k}\beta_{q-m}(u)}{(k)!(m-k)!(q-m)!} \times \\ \times \int_0^T \int_{\mathbb{S}^2} H_{q-m}(Z_t(x)) H_k(\widetilde{\partial}_{1;x} Z_t(x)) H_{m-k}(\widetilde{\partial}_{2;x} Z_t(x)) \, dx \, dt$$

$$\mathcal{C}_T(u)[2] = k\phi(u) \sum_{\ell} c_\ell \left\{ (u^2 - 1) + \frac{\lambda_\ell/2}{\sigma_1^2} \right\} \int_0^T \int_{\mathbb{S}^2} H_2(\widehat{Z}_\ell(x,t)) dx dt$$

- $\mathcal{C}_T(u^\star)[2] \equiv 0 \iff \#\mathcal{I}^\star = 1$ and $u^\star = \pm\sqrt{1 - \frac{\lambda_\ell}{2\sigma_1^2}}$

Non-universality and chaoses for $\mathcal{M}_T(u)$

Marinucci, Rossi, V. [AAP21]

chaos	$u \neq 0$	$u = 0$	asympt. distrib.
I	$\beta_0 < 2\beta_{\ell^\star} \wedge 1$ $\mathrm{Var} \approx T^{2-\beta_0}$	$\beta_0 < 3\beta_{\ell^\star} \wedge 1$ $\mathrm{Var} \approx T^{2-\beta_0}$	Gaussian
II	$2\beta_{\ell^\star} < \beta_0 \wedge 1$ $\mathrm{Var} \approx T^{2-2\beta_{\ell^\star}}$	never (vanishes)	*composite* Rosenblatt
III	never	$3\beta_{\ell^\star} < \beta_0 \wedge 1$ $\mathrm{Var} \approx T^{2-3\beta_{\ell^\star}}$	*composite* Rosenblatt
all	$2\beta_{\ell^\star} > \beta_0 > 1$ $\mathrm{Var} \approx T$	$3\beta_{\ell^\star} > \beta_0 > 1$ $\mathrm{Var} \approx T$	Gaussian

- Berry's cancellation at $u = 0$

Non-universality and chaoses for $\mathcal{C}_T(u)$

Marinucci, Rossi, V. [AHL24]

chaos	$\#\mathcal{I}^\star > 1$	$u = u^\star,\ \#\mathcal{I}^\star = 1$	asympt. distrib.
I	$\beta_0 < 2\beta_{\ell^\star} \wedge 1$ $\mathrm{Var} \approx T^{2-\beta_0}$	$\beta_0 < 3\beta_{\ell^\star} \wedge 1$ $\mathrm{Var} \approx T^{2-\beta_0}$	Gaussian
II	$2\beta_{\ell^\star} < \beta_0 \wedge 1$ $\mathrm{Var} \approx T^{2-2\beta_{\ell^\star}}$	never (vanishes)	*composite* Rosenblatt
III	never	$3\beta_{\ell^\star} < \beta_0 \wedge 1$ $\mathrm{Var} \approx T^{2-3\beta_{\ell^\star}}$	*composite* Rosenblatt
all	$2\beta_{\ell^\star} > \beta_0 > 1$ $\mathrm{Var} \approx T$	$3\beta_{\ell^\star} > \beta_0 > 1$ $\mathrm{Var} \approx T$	Gaussian

- Berry's cancellation at $u^\star = \pm\sqrt{1 - \frac{\lambda_\ell}{2\sigma_1^2}}$ if $\#\mathcal{I}^\star = 1$

Memory of the field vs memory of the statistics

- $Z = \sum_{\ell \geq 0} Z_\ell$
  - the smaller is β_ℓ the larger is the memory of Z_ℓ
  - $\sum_{\ell \in \mathcal{I}^\star} Z_\ell$ dominates the behavior of Z with memory $\beta_{\ell^\star}$
- $\mathcal{M}_T(u) = \sum_{q \geq 1} \mathcal{M}_T(u)[q] = \sum_{q \geq 1} c_q^u \int H_q(Z)$
  - $\int H_q\Big(\sum_{\ell \in \mathcal{I}^\star} Z_\ell\Big)$ dominates the behavior of $\mathcal{M}_T(u)[q]$
  - diagram formula: $\mathcal{M}_T(u)[q]$ has memory parameter $q\beta_{\ell^\star}$

Comments

- Phase transitions induced by the diagram formula:

$$\mathrm{Cov}\left(H_q(Z_t(x)), H_q(Z_s(y))\right) \approx \mathrm{Cov}\left(Z_t(x), Z_s(y)\right)^q$$

- **long memory:**
  non-summable covariance but $\exists\, q_0$ such that $(\text{covariance})^q$ is summable $\forall q \geq q_0 \implies$ higher chaoses do not contribute to the asymptotic behavior of $\mathcal{M}_T(u)$

- **short memory:**
  summable covariance $\implies$ $(\text{covariance})^q$ still summable
  $\implies$ all the chaoses contribute to the asymptotic behavior of $\mathcal{M}_T(u)$

A look at some statistical questions

Spherical AutoRegressive Process (SPHAR)

- Introduced in Caponera, Marinucci [AoS2021]:

$$Z_t(x) = \Phi_1 Z_{t-1}(x) + \cdots + \Phi_p Z_{t-p}(x) + W_t(x)$$

- W Gaussian spherical white noise
- $(\Phi_i)_{i=1,\ldots,p}$ isotropic kernel operators
- model estimation via OLS [AoS21] or LASSO [SPA21].

Statistical testing in Caponera, Marinucci, V. (2023)

- Adding a deterministic trend:

$$Z_t(x) = \mu_t(x) + X_t(x)$$

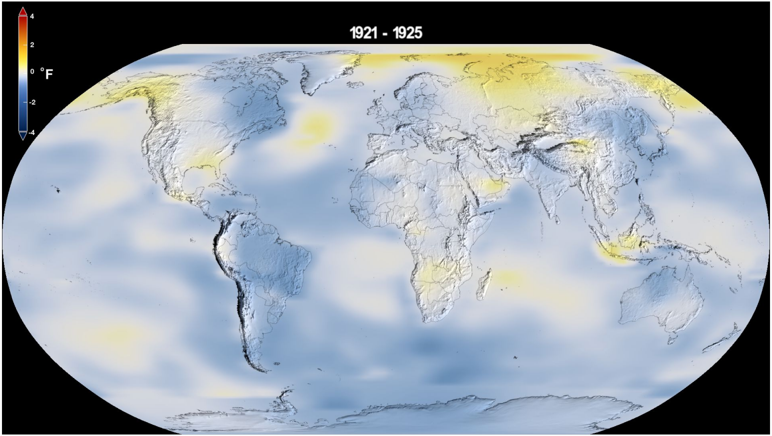
- and testing

$$H_0 : \mu_t(x) \equiv \mu(x) \qquad \text{vs} \qquad H_1 : \mu_t(x) \sim t^{\alpha}, \quad t \to \infty$$

- More precisely: harmonic/spectral methods to introduce a growing algebraic trend which can be different from multipole to multipole

$$\mu_t(x) = \sum_{\ell,m} \mu_{\ell,m}(t) Y_{\ell,m}(x) \qquad H_1 : \mu_{\ell,m}(t) \sim t^{\alpha_\ell}, \quad t \to \infty$$

Application to temperature dataset built starting from the monthly averages of the surface air temperature from 1948 to 2020



Future perspectives

- Other stationary/non-stationary geometric processes
- Topological processes (non-local): Betti numbers like number of connected components (how long does a connected component last?). See the works by Nazarov, Sodin, Beliaev .....
- Cointegration between different geometric functionals
- Using geometry to build statistical testing

GRAZIE!

References

[AAP21] D. Marinucci, M. Rossi, A. Vidotto. Non-Universal Fluctuations of the Empirical Measure for Isotropic Stationary Fields on $\mathbb{S}^2 \times \mathbb{R}$, *The Annals of Applied Probability*, vol. 31, no. 5, 2311-2349, 2021

[AHL24] D. Marinucci, M. Rossi, A. Vidotto. Fluctuations of Level Curves for Time-Dependent Spherical Random Fields, preprint arXiv:2207.13028, *to appear on Annales Henri Lebesgue*, 2024+

[AoS21] A. Caponera, D. Marinucci. Asymptotics for spherical functional autoregressions. *Annals of Statistics*, vol. 49, no. 1, pp. 346-369, 2021

[SPA21] A. Caponera, C. Durastanti, A. Vidotto. LASSO estimation for spherical autoregressive processes, *Stochastic Processes and their Applications*, vol. 137, 167-199, 2021

[pp23] A. Caponera, D. Marinucci, A. Vidotto. MultiScale CUSUM Tests for Time-Dependent Spherical Random Fields, preprint `arXiv:2305.01392`, 2023