Persistence of asymptotic variance under transports

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1. Random measures and point processes

Setting

$(\Omega, \mathcal{A}, \mathbb{P})$ is a fixed probability space.

Definition

(i) A random measure on $\mathbb{R}^d$ is a random element Φ in the space **M** of all locally finite measures on $\mathbb{R}^d$ equipped with a suitable σ-field.

(ii) A point process is a random measure Φ which is integer-valued on bounded Borel sets. Then $\Phi(B)$ is interpreted as the (random) number of points of Φ in a Borel set $B \subset \mathbb{R}^d$.

Definition

A random measure Φ on $\mathbb{R}^d$ is stationary if $\theta_x \Phi \overset{d}{=} \Phi$ for all $x \in \mathbb{R}^d$, where $\theta_x \Phi := \Phi(\cdot + x)$. It is ergodic if $\mathbb{P}(\Phi \in A) \in \{0, 1\}$ for all translation invariant measurable $A \subset \mathbf{M}$. The intensity of a stationary random measure Φ is defined by

$$\gamma \equiv \gamma_\Phi := \mathbb{E}[\Phi([0,1]^d)].$$

If Φ is a point process, then γ_Φ is also known as number density.

Remark

If Φ is a stationary random measure then

$$\mathbb{E}[\Phi(B)] = \gamma_\Phi \lambda_d(B), \quad B \in \mathcal{B}^d,$$

where λ_d denotes Lebesgue measure on $\mathbb{R}^d$.

Definition

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be measurable and define

$$\Phi(f) := \int f(x)\,\Phi(dx).$$

If Φ is a point process (without multiplicities), then

$$\Phi(f) = \sum_{x\in\Phi} f(x).$$

The random variable $\Phi(f)$ is said to be a linear functional of Φ.

Theorem (Campbell's formula)

The expectation of $\Phi(f)$ is given by

$$\mathbb{E}[\Phi(f)] = \gamma \int f(x)\,dx.$$

Setting

Φ is a stationary, locally square-integrable random measure on $\mathbb{R}^d$ with positive (and finite) intensity γ_Φ.

Definition

The measure α_Φ, defined by

$$\alpha_\Phi(B) := \mathbb{E} \iint \mathbf{1}\{x \in [0,1]^d, y - x \in B\}\Phi(dx)\Phi(dy), \quad B \in \mathcal{B}^d,$$

is called the reduced second moment measure of Φ.

Fact

The measure α_Φ is locally finite and reflection symmetric.

Remark

If Φ is a point process, then it is often convenient to consider the reduced second factorial moment measure $\alpha^!_\Phi$ of Φ, defined for measurable $B \subset \mathbb{R}^d$ by

$$\alpha^!_\Phi(B) := \mathbb{E} \int \mathbf{1}\{x \in [0,1]^d, y - x \in B\}\, \Phi^{(2)}(d(x,y))$$
$$= \mathbb{E} \sum_{x,y\in\Phi, x\neq y} \mathbf{1}\{x \in [0,1]^d, y - x \in B\},$$

where $\Phi^{(2)}$ is the second factorial measure of Φ. We then have

$$\alpha_\Phi = \alpha^!_\Phi + \gamma_\Phi \delta_0.$$

Definition

Assume that Φ is a point process such that $\alpha^!_\Phi$ has a density γ_2. Then the function $g_2 := \gamma_\Phi^{-2}\gamma_2$ is called pair correlation function of Φ. Heuristically,

$$g_2(x-y)dxdy = \gamma_\Phi^{-2}\mathbb{E}[\Phi(dx)\Phi(dy)], \quad x \neq y.$$

Definition

The covariance measure β_Φ of Φ is the signed measure

$$\beta_\Phi := \alpha_\Phi - \gamma_\Phi^2 \lambda_d.$$

Remark

The covariance measure β_Φ is well-defined and finite on bounded Borel sets. Its total variation measure $|\beta_\Phi|$ is a well-defined locally finite measure on $\mathbb{R}^d$.

Remark

Assume that Φ is a point process. Then

$$\beta_\Phi = \gamma_\Phi \delta_0 + \alpha_2^! - \gamma_\Phi^2 \lambda_d.$$

If the pair correlation function g_2 exists, then

$$\beta_\Phi = \gamma_\Phi \delta_0 + \gamma_\Phi^2 (g_2 - 1) \cdot \lambda_d.$$

Theorem

Let $f, g \colon \mathbb{R}^d \to \mathbb{R}$ be bounded measurable functions with bounded support. Then

$$\mathbb{C}\mathrm{ov}[\Phi(f), \Phi(g)] = \int f(x)g(x+y)\,\beta_\Phi(dy)\,dx$$
$$= \int f \star g(y)\,\beta_\Phi(dy),$$

where

$$(f \star g)(y) := \int f(x+y)g(x)\,dx, \quad y \in \mathbb{R}^d,$$

is the tilted convolution of f and g.

Corollary

The measure β_Φ is positive semidefinite, that is

$$\int (f \star f)(y)\, \beta_\Phi(dy) \geq 0$$

for all bounded measurable $f\colon \mathbb{R}^d \to \mathbb{R}$ with bounded support.

Corollary

Let $W \subset \mathbb{R}^d$ be bounded and measurable. Then

$$\mathbb{V}\mathrm{ar}[\Phi(W)] = \int \lambda_d(W \cap (W + x))\, \beta_\Phi(dx).$$

If Φ has a pair correlation function, then

$$\mathbb{V}\mathrm{ar}[\Phi(W)] = \gamma_\Phi \lambda_d(W) + \gamma_\Phi^2 \int \lambda_d(W \cap (W + x))(g_2(x) - 1)\, dx.$$

2. The Bartlett spectral measure

Definition

The Fourier transform of $f \in L^1(\lambda^d)$ is defined by

$$\hat{f}(k) := \int f(x) e^{-\mathrm{i}kx}\, dx, \quad k \in \mathbb{R}^d.$$

Definition

There is a locally finite measure $\hat{\beta}_\Phi$ on $\mathbb{R}^d$ such that

$$\int f \star g(x)\, \beta_\Phi(dx) = \frac{1}{(2\pi)^d} \int \hat{f}(k)\hat{g}(-k)\, \hat{\beta}_\Phi(dk),$$

for all bounded measurable $f, g\colon \mathbb{R}^d \to \mathbb{R}$ with bounded support; see e.g. Berg and Forst '75. This is the Bartlett spectral measure (diffraction measure) of Φ; see Brémaud '20.

Definition

Assume that Φ is a point process with a pair correlation function g_2 such that $h_2 := g_2 - 1$ is integrable. The structure factor of Φ is the function $S_\Phi : \mathbb{R}^d \to \mathbb{R}$ defined

$$S_\Phi(k) := 1 + \gamma_\Phi \hat{h}_2(k), \quad k \in \mathbb{R}^d.$$

This definition can be extended to random measures with $|\beta_\Phi|(\mathbb{R}^d) < \infty$:

$$S_\Phi(k) := \gamma_\Phi^{-1} \int e^{-\mathrm{i}kx}\, \beta_\Phi(dx), \quad k \in \mathbb{R}^d,$$

Remark

If $|\beta_\Phi|(\mathbb{R}^d) < \infty$ then

$$\hat{\beta}_\Phi(dk) = \gamma_\Phi S_\Phi(k)\, dk.$$

Theorem

Assume that Φ is a point process with a pair correlation function g_2 such that $g_2 - 1$ is integrable. Then

$$\gamma_\Phi S_\Phi(k) = \lim_{r\to\infty} \lambda_d(rW)^{-1}\mathbb{E}\Big|\sum_{x\in\Phi\cap rW} e^{-\mathrm{i}kx}\Big|^2, \quad k \neq 0.$$

Theorem

Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be bounded measurable functions with bounded support. Then

$$\mathbb{C}\mathrm{ov}[\Phi(f), \Phi(g)] = \frac{1}{(2\pi)^d} \int \hat{f}(k)\hat{g}(-k)\, \hat{\beta}_\Phi(dk).$$

In particular,

$$\mathbb{V}\mathrm{ar}[\Phi(f)] = \frac{1}{(2\pi)^d} \int |\hat{f}(k)|^2\, \beta_\Phi(dk).$$

Example

Assume that Φ is a Poisson process. Then $\alpha^!_\Phi = \gamma^2_\Phi \lambda_d$, $g_2 \equiv 1$, $\beta_\Phi = \gamma_\Phi \delta_0$, and $\hat{\beta}_\Phi = \gamma_\Phi \lambda_d$.

Example (Stationary lattice)

Assume that

$$\Phi = \sum_{x \in \mathbb{L}} \delta_{U+x},$$

where $\mathbb{L}$ is a lattice (for instance $\mathbb{L} = \mathbb{Z}^d$) and U is uniformly distributed on the unit cell. Assume (for simplicity) that the fundamental region has Lebesgue measure 1. Then $\gamma_\Phi = 1$ and

$$\beta_\Phi = \sum_{x \in \mathbb{L}} \delta_x - \lambda_d.$$

The Bartlett spectral measure is given by $\hat{\beta}_\Phi = \sum_{x \in \mathbb{L}^* \setminus \{0\}} \delta_x$, where $\mathbb{L}^*$ is the dual lattice of $\mathbb{L}$.

Example (Independently perturbed stationary lattice)

Assume that

$$\Phi = \sum_{x\in\mathbb{L}} \delta_{U+x+V_x},$$

where U is as above and the V_x are iid such that the distribution $\mathbb{Q}^*$ of $V_x - V_y$ for $x \neq y$ does not charge $\mathbb{L}\setminus\{0\}$. Then

$$\alpha^!_\Phi = \sum_{x\in\mathbb{L}\setminus\{0\}} \int \mathbf{1}\{x+w\in\cdot\}\,\mathbb{Q}^*(dw) = (\mathbb{L}\setminus\{0\}) * \mathbb{Q}^*.$$

where $\mathbb{Q}^*$ is the distribution of $V_x - V_0$ for $x \neq 0$. We have

$$\hat{\beta}_\Phi = \lambda_d - |\hat{\mathbb{Q}}|^2 \cdot \lambda_d + |\hat{\mathbb{Q}}|^2 \cdot (\mathbb{L}^* - \delta_0),$$

where $\hat{\mathbb{Q}}(k) := \int e^{-i\langle k,x\rangle}\,\mathbb{Q}(dx)$. Note that the third term vanishes if $\hat{\mathbb{Q}}(k)$ vanishes on $\mathbb{L}^*$ (cloaking); see Klatt, Kim, Torquato '20.

3. Hyperuniformity

Definition

Let W be a convex bounded set containing 0 in its interior. Then Φ is said to be hyperuniform w.r.t. W if

$$\lim_{r\to\infty} \frac{\mathbb{V}\mathrm{ar}[\Phi(rW)]}{\lambda_d(rW)} = 0.$$

If W is the unit ball B_1 then $rW =: B_r$ and Φ is simply called hyperuniform.

Theorem

Assume that $|\beta_\Phi|(\mathbb{R}^d) < \infty$. *Then*

$$\lim_{r\to\infty} \frac{\mathbb{V}\mathrm{ar}[\Phi(rW)]}{\lambda_d(rW)} = \beta_\Phi(\mathbb{R}^d) \in [0,\infty].$$

In particular Φ *is then hyperuniform if and only if* $\beta_\Phi(\mathbb{R}^d) = 0$.

Proof. We have

$$\frac{\mathbb{V}\mathrm{ar}[\Phi(rW)]}{\lambda_d(rW)} = \int \frac{\lambda_d(rW \cap (rW + y))}{\lambda_d(rW)} \,\beta_\Phi(dy)$$

and

$$\lim_{r\to\infty} \frac{\lambda_d(rW \cap (rW + y))}{\lambda_d(rW)} = 1, \quad y \in \mathbb{R}^d.$$

Remark

If Φ has a pair correlation function g_2 then β_Φ has finite total variation if and only if

$$\int |g_2(x) - 1|\, dx < \infty.$$

In this case hyperuniformity is equivalent to

$$1 + \gamma_\Phi \int (g_2(x) - 1)\, dx = 0.$$

Corollary

Assume that β_Φ has finite total variation. Then

$$\lim_{r\to\infty} \frac{\mathbb{V}\mathrm{ar}[\Phi(rW)]}{\lambda_d(rW)} = \gamma_\Phi S_\Phi(0).$$

In particular Φ is then hyperuniform if and only if $S(0) = 0$.

Theorem (Björklund and Hartnick '24)

Assume that Φ is hyperuniform w.r.t. some W. Then Φ is spectrally hyperuniform, that is

$$\lim_{\varepsilon\to 0} \lambda_d(B_\varepsilon)^{-1}\hat{\beta}(B_\varepsilon) = 0.$$

If, conversely, Φ is spectrally hyperuniform, then Φ is hyperuniform w.r.t. each Fourier smooth convex body W.

Theorem (Torquato '18)

Assume that β_Φ has finite total variation and that Φ is hyperuniform. Assume also, that there exists $p > 0$ such that the structure factor satisfies

$$|S(k)| \sim \|k\|^p, \quad \textit{as } k \to 0.$$

Then $\sigma_r := \mathbb{V}\mathrm{ar}[\Phi(B_r)]$ satisfies for $r \to \infty$:

(i) *If $p > 1$ then $\sigma_r \sim r^{d-1}$.*

(ii) *If $p = 1$ then $\sigma_r \sim (\log r) r^{d-1}$.*

(iii) *If $p < 1$ then $\sigma_r \sim r^{d-p}$.*

Proof. For each $r > 0$ the Fourier transform of the ball $B_r := B(0, r)$ is given by

$$\int_{B_r} e^{\mathrm{i}\langle k,x\rangle}\, dx = \left(\frac{2\pi r}{\|k\|}\right)^{d/2} J_{d/2}(\|k\| r), \quad k \in \mathbb{R}^d \setminus \{0\},$$

where, for $a > -1/2$, the Bessel function $J_a \colon [0, \infty) \to \mathbb{R}$ is given by

$$J_a(r) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + a + 1)} \left(\frac{r}{2}\right)^{2m+a}, \quad r \geq 0.$$

By the previous formulas,

$$\mathbb{V}\mathrm{ar}[\Phi(B_r)] = \frac{\gamma r^d}{(2\pi)^d} \int \frac{J_{d/2}(r\|k\|)^2}{\|k\|^d} S(k)\, dk$$
$$= \frac{\gamma r^d}{(2\pi)^d} \int \frac{J_{d/2}(\|x\|)^2}{\|x\|^d} S(x/r)\, dx.$$

Splitting the integrals according to $\|x\| \le r$ and $\|x\| > r$, the result follows from

$$J_{d/2}(t) \sim t^{-1/2}, \quad \text{as } t \to \infty.$$

Example

Let $K\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a continuous, and positive semidefinite function. Assume that Φ is a determinantal process with kernel K satisfying

$$|K(x,y)| = |K(0,y-x)| = |K(0,x-y)|, \quad x,y \in \mathbb{R}^d.$$

Then $\gamma = K(0,0)$ and

$$\alpha^!_\Phi(dx) = (\gamma^2 - K(0,x)^2)\, dx.$$

Therefore Φ has covariance measure

$$\beta_\Phi(dx) = \gamma\delta_0(dx) - |K(0,x)|^2\, dx$$

and structure factor $S = 1 - \gamma^{-1}\hat{h}$, where $h(x) := |K(0,x)|^2$. If h is integrable, then hyperuniformity is equivalent to

$$K(0,0) = \int |K(0,x)|^2\, dx.$$

Example (Ginibre ensemble)

Consider $\mathbb{C} \equiv \mathbb{R}^2$ and the kernel

$$K(x,y) := \pi^{-1} e^{x\bar{y}} e^{-|x|^2/2-|y|^2/2}, \quad x,y \in \mathbb{C}.$$

A determinantal point process Φ with this kernel describes the (weak limit of the) eigenvalues of random complex Gaussian matrices. The structure factor is given by

$$S(k) = 1 - e^{-|k|^2/4}$$

Hence Φ is hyperuniform (of class (i)); see Ghosh and Lebovitz '17.

Definition (Lotz, Klatt '26)

Let $p \in [-d, \infty)$. Then Φ is called *p-uniform* if, as $\varepsilon \to o$,

$$\varepsilon^{-d} \hat{\beta}(B_\varepsilon) = O(\varepsilon^p)$$

and *beyond p-uniform* if

$$\varepsilon^{-d} \hat{\beta}(B_\varepsilon) = o(\varepsilon^p).$$

Remarks

- If Φ has a structure factor satisfying

$$S_\Phi(k) = a\|k\|^p + o(\|k\|^p)$$

  as $k \to 0$, for some $a \geq 0$, then Φ is p-uniform. If $a = 0$, then Φ is beyond p-uniform.
- Hyperuniformity means beyond 0-uniformity.
- The case $\varepsilon^{-d}\hat{\beta}(B_\varepsilon) \sim \varepsilon^p$ for $p < 0$ describes hyperfluctuation.
- Φ is always $(-d)$-uniform and beyond $(-d)$-uniform iff it is pseudo-ergodic.

Definition

An integrable $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to be Fourier smooth with exponent $q \geq -d$ if

$$|\hat{f}(k)| \leq c(1 + \|k\|)^{-(d+q)/2}, \quad k \in \mathbb{R}^d,$$

for some $c > 0$. In the case $q \leq 0$ some extra condition is required.

Theorem (Lachiéze-Rey '25, Lotz and Klatt '26)

Suppose that Φ is p-uniform and let $f \in L^1(\mathbb{R}^d)$ be Fourier smooth with exponent $q > p$. Then

$$r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(f_r)] = O(r^{-p}),$$

where $f_r(x) := f(x/r)$. The same holds with o instead of O if Φ is beyond p-uniform. The converse holds if $\int f(x)\,dx \neq 0$.

Example

Assume that $f = \mathbf{1}_{B_1}$ is the indicator of the unit ball. This f is Fourier smooth with exponent 1. If Φ is p-uniform for some $p < 1$, then $r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(f_r)] = O(r^{-p})$ which corresponds to class (iii).

Theorem (Lachiéze-Rey '25, Lotz and Klatt '26)

Suppose that Φ is p-uniform and let $f \in L^1(\mathbb{R}^d)$ be Fourier smooth with exponent q. If $q < p$. Then

$$r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(f_r)] = O(r^{-q}).$$

If $p = q$, then $r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(f_r)] = O(\log(r)r^{-p})$.

Example

Assume that $f = \mathbf{1}_{B_1}$. If Φ is p-uniform for some $p > 1$, then $r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(f_r)] = O(r^{-1})$ which corresponds to class (i).

Theorem (Beck '87, Lachiéze-Rey '25, Lotz and Klatt '26)

If $r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(B_r)] = O(r^{-1})$, then Φ is beyond 1-uniform. If

$$r^{-d}\,\mathbb{V}\mathrm{ar}[\Phi(B_r)] = o(r^{-1}).$$

then Φ is a multiple of Lebesgue measure.

4. Allocations

Setting

- All random measures and kernels are assumed to be jointly stationary in a suitable sense.
- $\mathbb{P}_0^\Phi$ denotes the Palm probability measure associated with Φ, while $\mathbb{P}_{0,y}^\Phi$, $y \in \mathbb{R}^d$, denote the two-point Palm probability measures associated with Φ.

Definition

An invariant allocation is a random mapping $\tau : \mathbb{R}^d \to \mathbb{R}^d$ which depends on the underlying randomness in a measurable and translation covariant way. Then $\{\tau(x) - x : x \in \mathbb{R}^d\}$ is also known as displacement field or pertubation.

Definition

Let Φ be a random measure and τ an allocation. Then τ transports the source source Φ to the destination

$$\tau\Phi := \int \mathbf{1}\{\tau(x) \in \cdot\}\, \Phi(dx).$$

If Φ is a point process without multiplicities, then

$$\tau\Phi = \sum_{x \in \Phi} \delta_{\tau(x)}.$$

Theorem

Let τ be an invariant allocation and define the mixing function

$$\kappa(y) := \left\| \mathbb{P}^{\Phi}_{0,y}((\tau(y) - y, \tau(0)) \in \cdot) - \mathbb{P}^{\Phi}_{0}(\tau(0) \in \cdot)^{\otimes 2} \right\|, \quad y \in \mathbb{R}^d,$$

where $\|\cdot\|$ denotes the total variation norm. If

$$\int \kappa(y)\, \alpha_{\Phi}(dy) < \infty,$$

then $\tau\Phi$ has the same asymptotic variance as Φ.

Remark

If Φ has a pair correlation function g satisfying $\int |g(x) - 1|\, dx < \infty$, then the assumption of the previous theorem is equivalent to $\int \kappa(y)g(y)\, dy < \infty$.

Example (Stationary perturbed lattice I)

Let $\Phi = \sum_{x\in\mathbb{Z}} \delta_{x+U}$ be the stationary lattice and assume that $\{\tau(x) - x : x \in \Phi\}$ is stationary and independent of U. Then the previous assumption simplifies to

$$\sum_{y\in\mathbb{Z}^d} \kappa(y) < \infty,$$

and $\kappa(y)$ is the β-mixing coefficient between the random variables $\tau(y) - y$ and $\tau(0)$; see also Flimmel '25.

Example (Stationary perturbed lattice II)

Let Φ be again the stationary lattice and assume that $\{\tau(x) - x : x \in \Phi\}$ is a stationary Gaussian random field (independent of U). If

$$\sum_{y \in \mathbb{Z}^d} \| \mathbb{C}\mathrm{ov}(\tau(y) - y, \tau(0)) \| < \infty,$$

then $\tau\Phi$ is hyperuniform.

Theorem (Dereudre, Flimmel, Huesmann, Leblé ’24)

Consider the stationary perturbed lattice Ψ.

(i) *Assume that $d \in \{1, 2\}$ and $\mathbb{E}[\|\tau(0)\|^d] < \infty$. Then Ψ is hyperuniform.*

(ii) *If $X \geq 0$ is a random variable with $\mathbb{E}[X^d] = \infty$, then there exists a displacement field with $\|\tau(0)\| \leq X$ such that the local number variance of Ψ is infinite.*

(iii) *Suppose $d \geq 3$. For each $\varepsilon > 0$ there exists a displacement field with $\|\tau(0)\| \leq \varepsilon$ such that $r^{-d}\,\mathbb{V}\mathrm{ar}[\Psi(B_r)] \to \infty$ as $r \to \infty$.*

Remark

In the case $d = 2$ hyperuniformity of a point process Ψ is very close to the existence of a stationary matching (a bijective allocation) between the stationary lattice and Ψ, such that the distance between a typical point and its matching partner has a finite second moment; see Lachiéze-Rey&Yogeshwaran '24 and Huesmann&Leblé '24.

5. Fair partitions

Definition

A stationary partition is a pair (Φ, τ) consisting of a stationary point process $\Phi \neq 0$ with finite intensity γ_Φ and an allocation τ, such that $\tau(x) \in \Phi$ for all $x \in \mathbb{R}^d$. The cells of this partition are given by

$$C^\tau(x) := \{y \in \mathbb{R}^d : \tau(y) = x\}, \quad x \in \Phi.$$

The partition is said to be fair (or balanced), if all cells have the same volume, that is

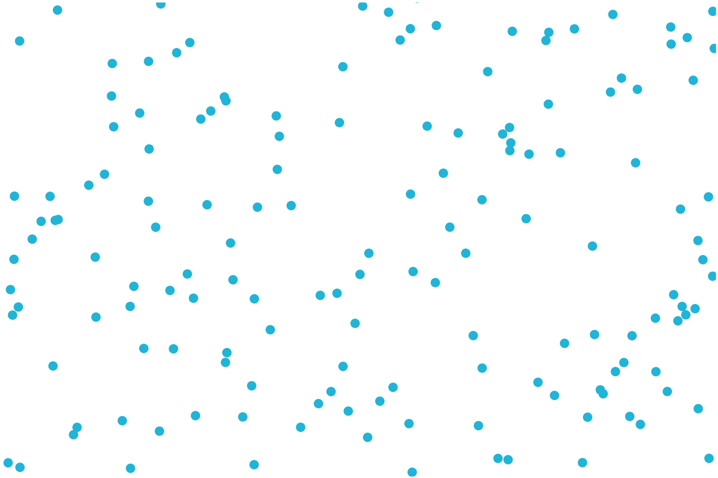
$$\mathbb{P}(\lambda_d(C^\tau(x)) = \gamma_\Phi^{-1} \text{ for all } x \in \Phi) = 1.$$

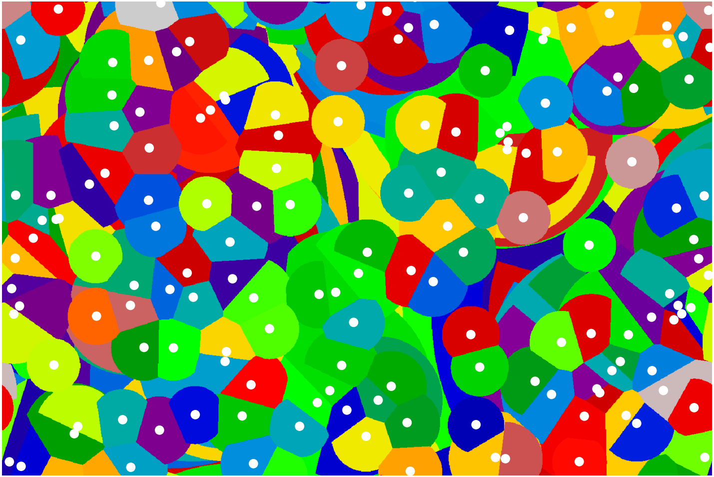
Theorem (KLLY '25)

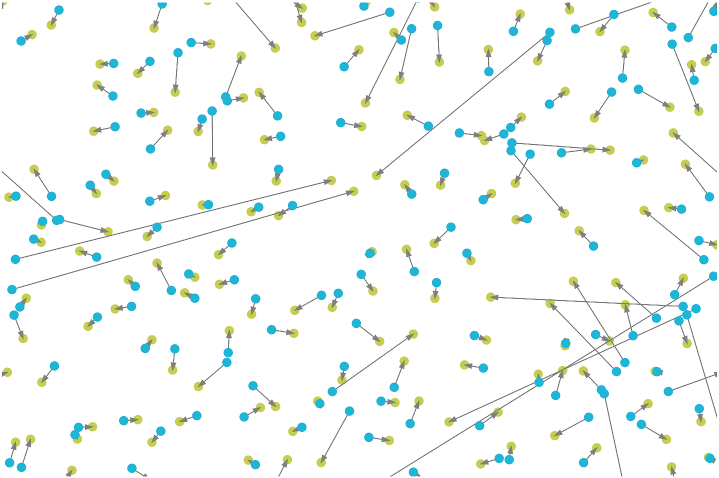
Suppose that (Φ, τ) is a fair partition. Let $Z(x)$, $x \in \Phi$, be random vectors in $\mathbb{R}^d$ which are, given (τ, Φ), conditionally independent and uniformly distributed on $C^\tau(x)$. Then the point process

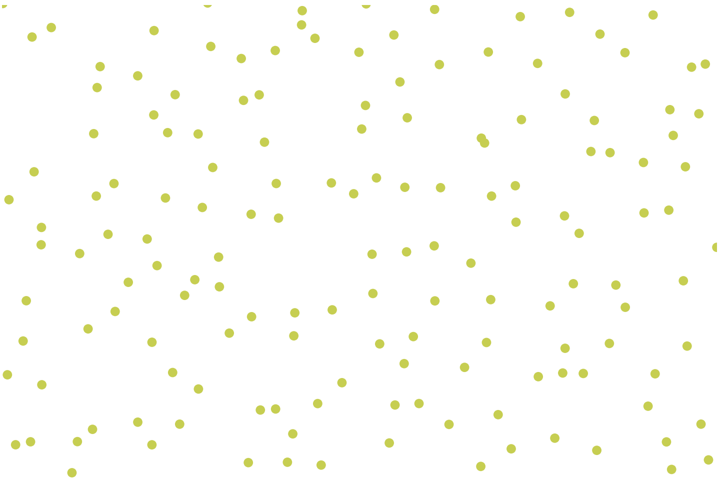
$$\Gamma := \sum_{x \in \Phi} \delta_{Z(x)}$$

is hyperuniform.









Remark

There exist hyperuniform purely discrete random measures, which are ergodic and isotropic, and whose support is dense in $\mathbb{R}^d$.

Theorem (KLLY '26)

Let Φ be a stationary Poisson process of non-zero intensity γ. For $k \in \mathbb{N}$, define Φ_k as the k-th iterate of Lloyd's algorithm starting with Φ. Then, for all $k \in \mathbb{N}$, we have that

$$\lim_{r\to\infty} \lambda_d(B_r)^{-1}\, \mathbb{V}\mathrm{ar}[\Phi_k(B_r)] = \gamma.$$

7. General transports

Definition

Let K be an (invariant) random probability kernel from $\mathbb{R}^d$ to $\mathbb{R}^d$. Then $K\Phi$ is the random measure on $\mathbb{R}^d$ defined by

$$K\Phi := \int K(x,\cdot)\,\Phi(dx).$$

We refer to K as a transport kernel transporting the source Φ to the destination $K\Phi$.

Remark

- We assume Φ and K to be jointly stationary. Therefore $K\Phi$ has the same intensity as Φ.
- If τ is an allocation, then δ_τ is a transport kernel.

Definition

Let K be a transport kernel. Define the mixing function

$$\kappa(y) := \left\| \mathbb{E}^{\Phi}_{0,y}[K^*_y \otimes K^*_0] - \left(\mathbb{E}^{\Phi}_0[K^*_0]\right)^{\otimes 2} \right\|, \quad y \in \mathbb{R}^d,$$

where $K^*_x(\omega, \cdot) := K(\theta_x \omega, 0, \cdot)$, $(\omega, x) \in \Omega \times \mathbb{R}^d$, denotes the relative transport kernel and $\theta_x \colon \Omega \to \Omega$ is the underlying shift operator.

Theorem (KLLY '25)

Assume that

$$\int \kappa(y)\, \alpha_\Phi(dy) < \infty.$$

Assume further, that either the convex body W is Fourier smooth or that β_Φ has finite total variation. Then Φ and $K\Phi$ have the same asymptotic variance w.r.t. W.

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