Locating holes in silence

Topological data analysis applied to the zeros of the short-time Fourier transform

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Time-frequency detection and reconstruction

The short-time Fourier transform[1]

Let $g : t \mapsto Ce^{-t^2/2}$ with $\|g\|_2 = 1$. For $f \in L^2(\mathbb{R})$, consider

$$V_g f(u, v) = \int f(t)\overline{g(t-u)}e^{-2\mathrm{i}\pi tv}dt = \langle f, M_v T_u g\rangle.$$

[1]Flandrin, 1998; Gröchenig, 2001.

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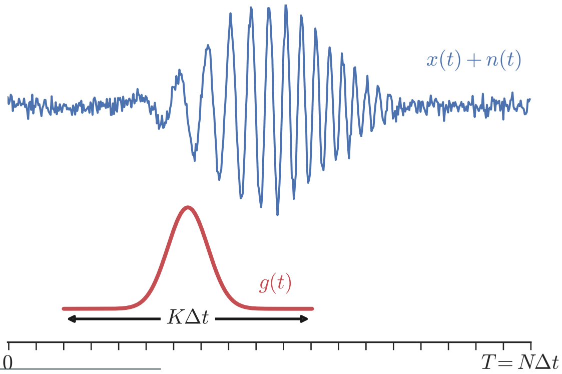
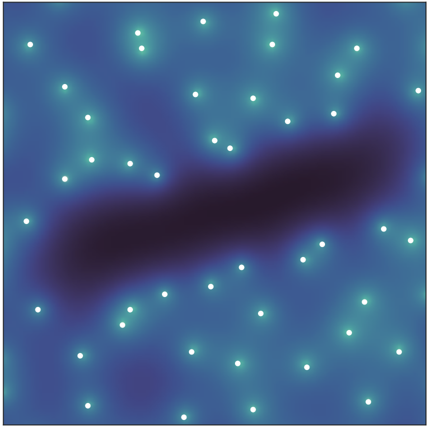
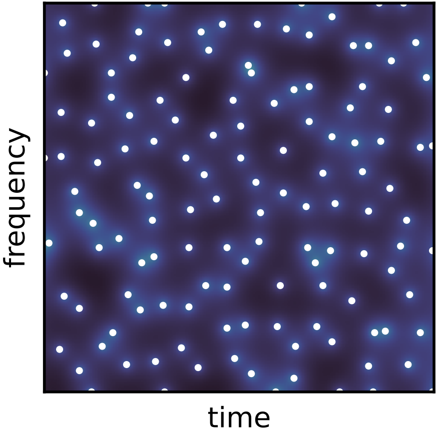


[1]Flandrin, 1998; Gröchenig, 2001.

The point process formed by the zeros of the STFT[2]





[2] Flandrin, 2015; Bardenet et al., 2020; Haimi et al., 2022.

Statistical signal processing with zeros of spectrograms

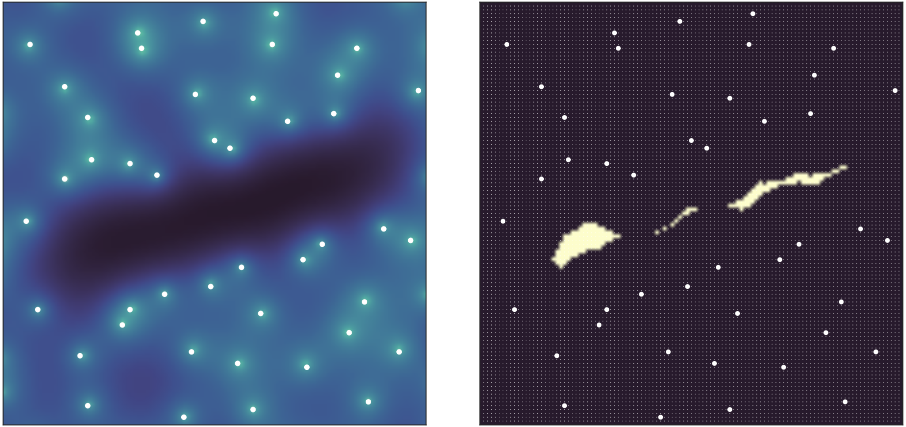
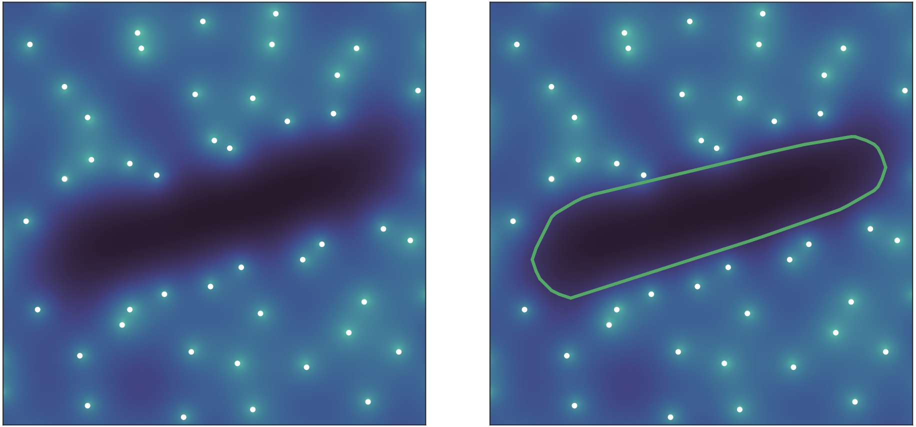


Figure 1: Taken from Bardenet et al., 2020

Statistical signal processing with zeros of spectrograms



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Statistical signal processing with zeros of spectrograms

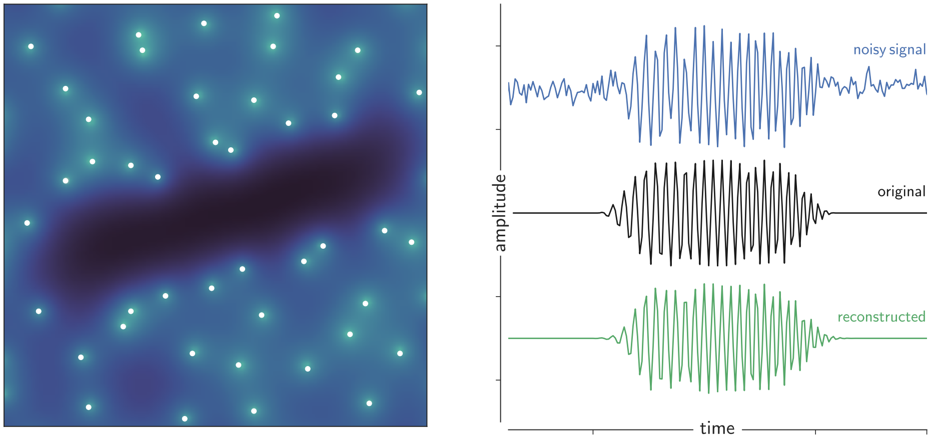


Figure 1: Taken from Bardenet et al., 2020

The original detection and reconstruction of Flandrin, 2015

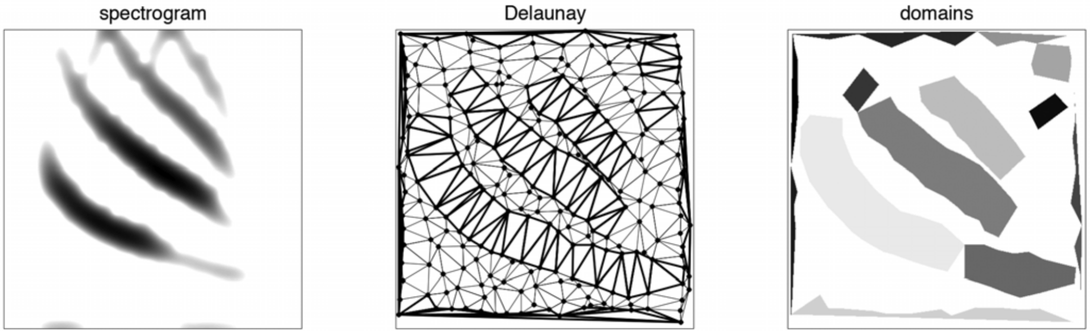
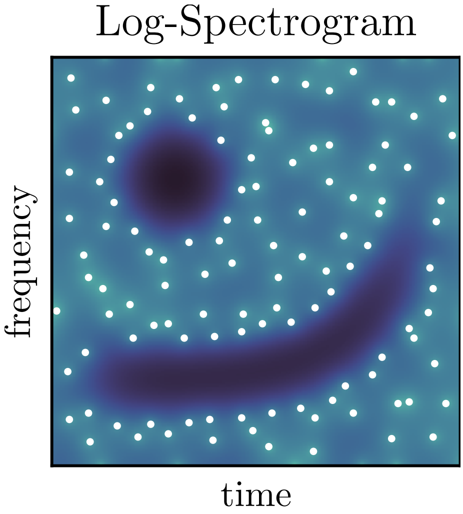
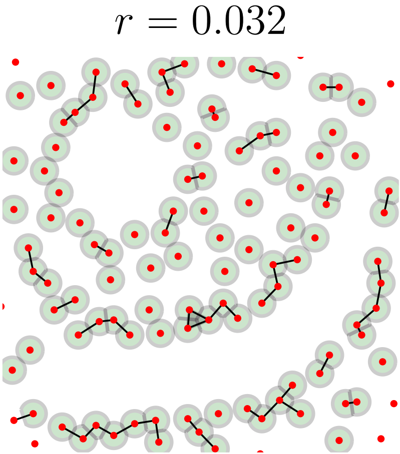


Figure 2: Taken from Flandrin, 2015

Tools from TDA

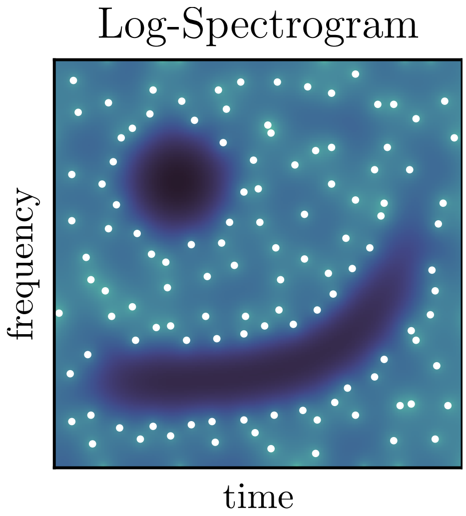
The α-complex[3]

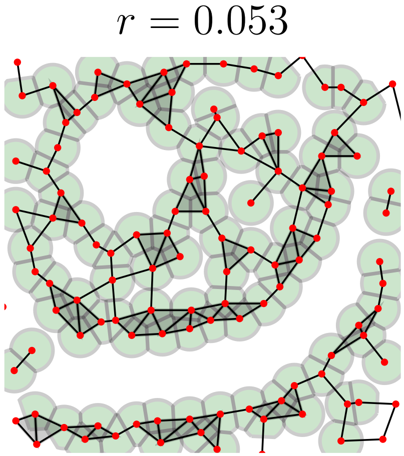




[3]Edelsbrunner et al., 2022.

The α-complex[3]





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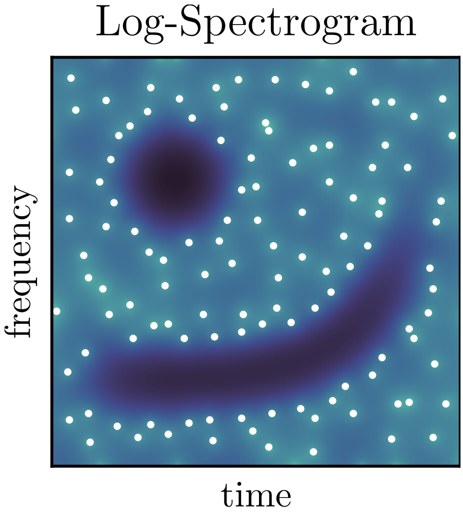

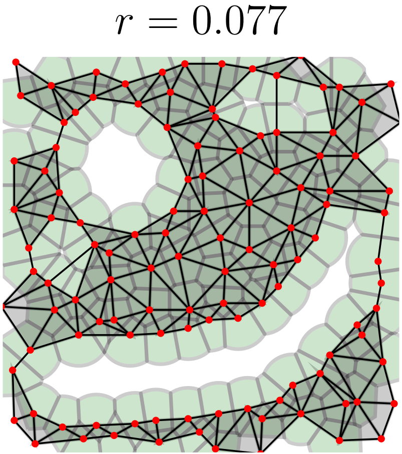

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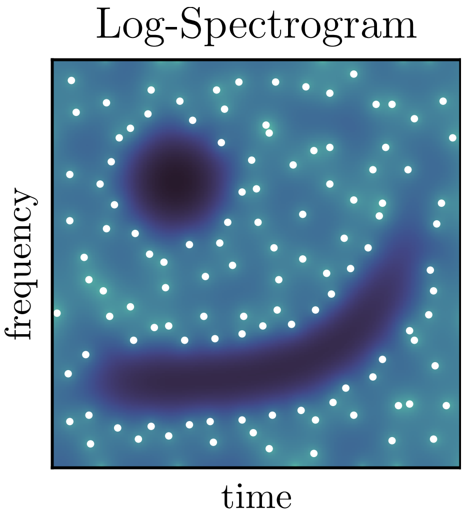
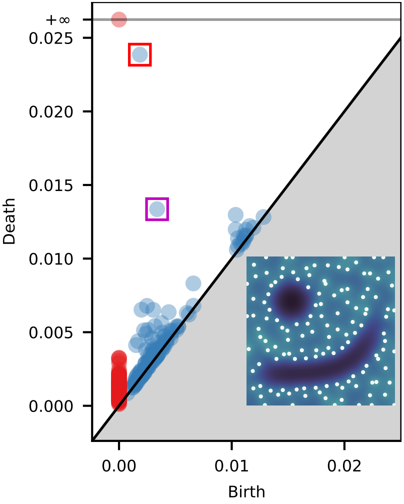
The persistence diagram[4]





[4] Edelsbrunner et al., 2022.

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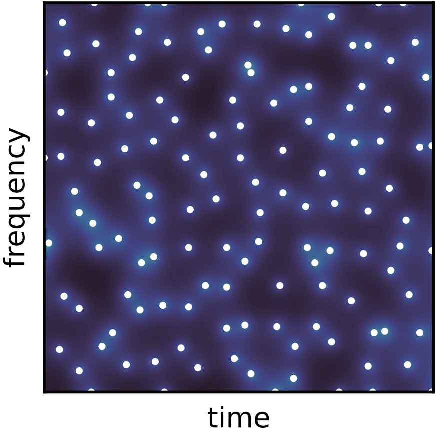
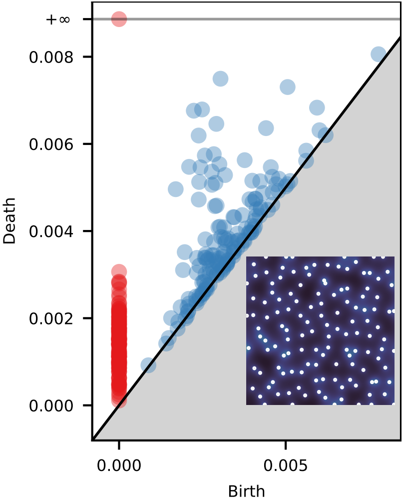
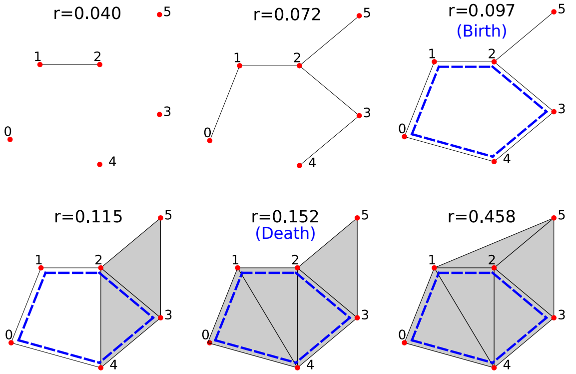




[4] Edelsbrunner et al., 2022.

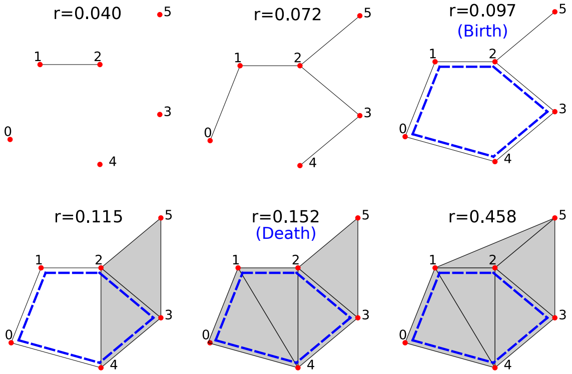
The persistence tree[5]



$$\{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0)\}$$

[5] Schweinhart, 2015.

The persistence tree[5]



$$\{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(12,0.040)\}$$

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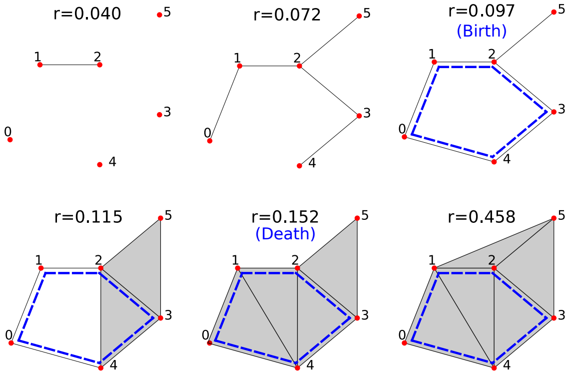

$$\{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(12,0.040),\cdots,(04,0.097),(35,0.111),(24,0.111),(235,0.115),(234,0.115)\}$$

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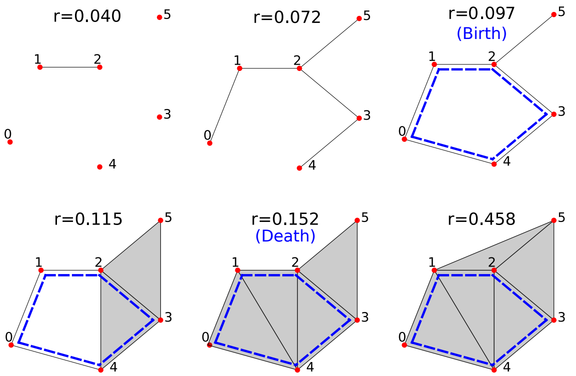
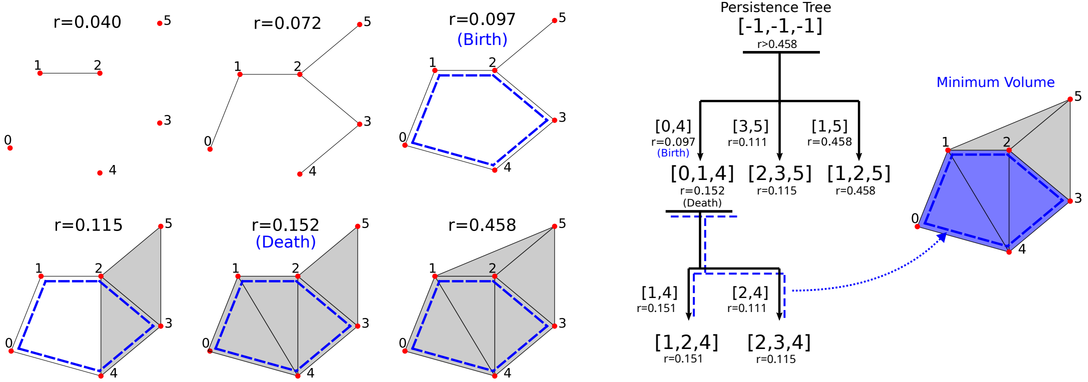
The persistence tree[5]



$$\{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(12,0.040),\ldots,(04,0.097),(35,0.111),(24,0.111),(235,0.115),(234,0.115),(14,0.151),(124,0.151),(014,0.152),(15,0.458),(125,0.458)\}$$

[5]Schweinhart, 2015.

The persistence tree[5]



$$\{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(12,0.040),\ldots,(04,0.097),(35,0.111),(24,0.111),(235,0.115),(234,0.115),(14,0.151),(124,0.151),(014,0.152),(15,0.458),\color{magenta}{(125,0.458)}\}$$

[5]Schweinhart, 2015.

Stable volumes[6]

For $\epsilon \geq 0$,

$$SV_\epsilon(\sigma_b, \sigma_d) \triangleq \{\sigma_d\} \cup \left(\bigcup_{\sigma \in C_\epsilon(\sigma_b, \sigma_d)} \mathrm{des}(\sigma, G) \right), \qquad (1)$$

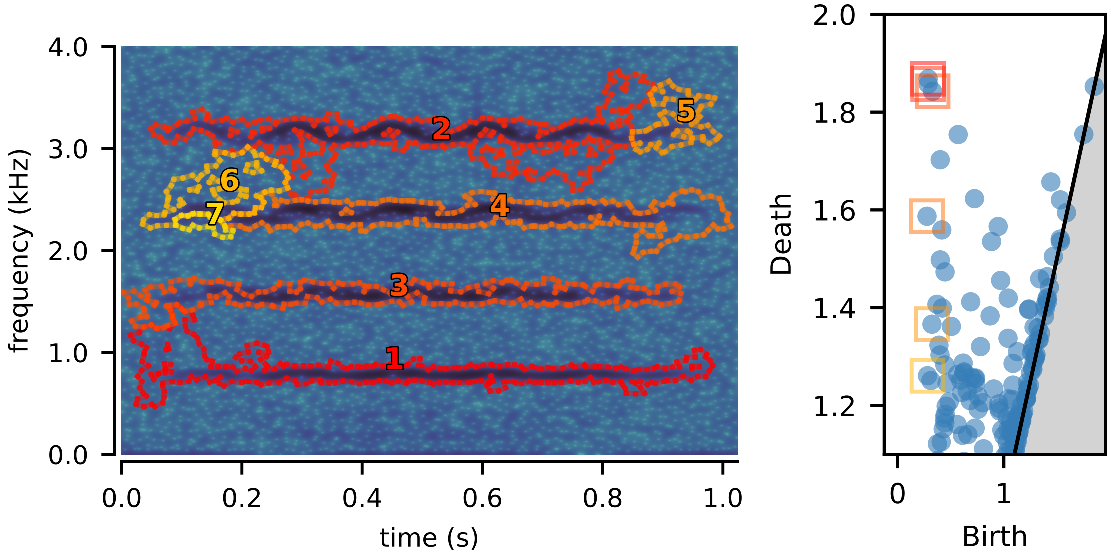
where

$$C_\epsilon(\sigma_b, \sigma_d) = \{\sigma \in X^{(2)} \mid \sigma_d \xrightarrow{r_k} \sigma \text{ and } r_k \geq (1+\epsilon)\, r_b\},$$

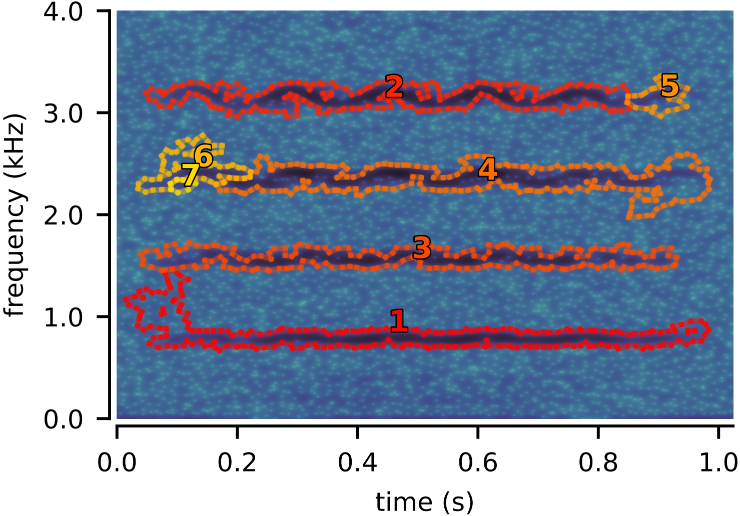
$X^{(2)}$ is the set of all 2-simplices.

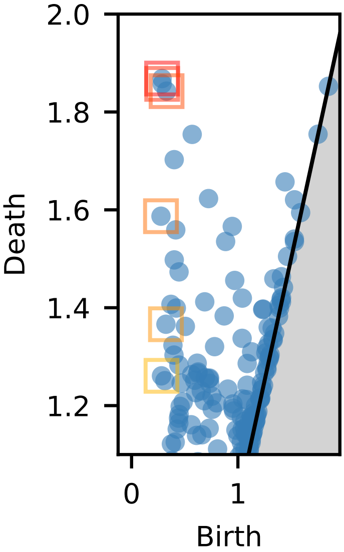
[6]Obayashi, 2023.

Minimum vs stable volumes



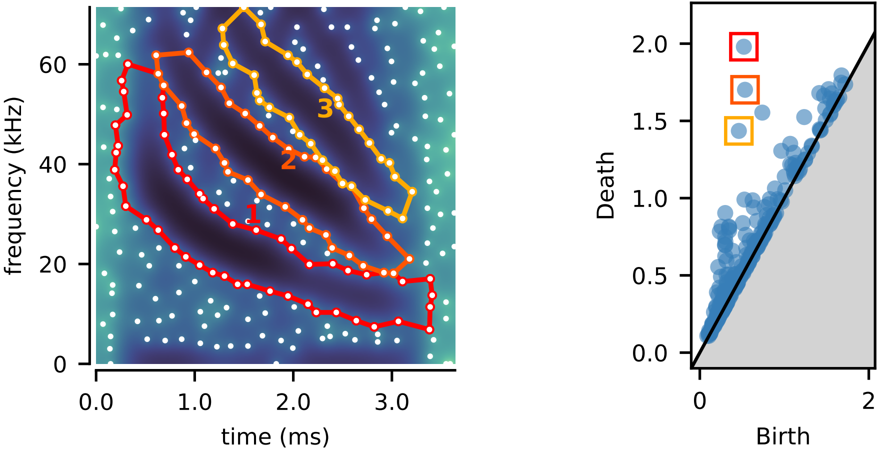
Minimum vs stable volumes



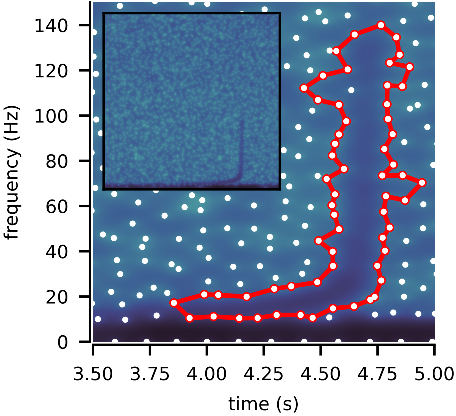


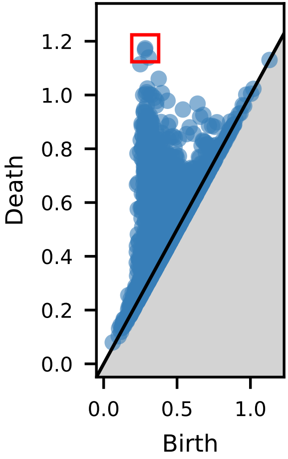
Experiments

The chirp of a bat

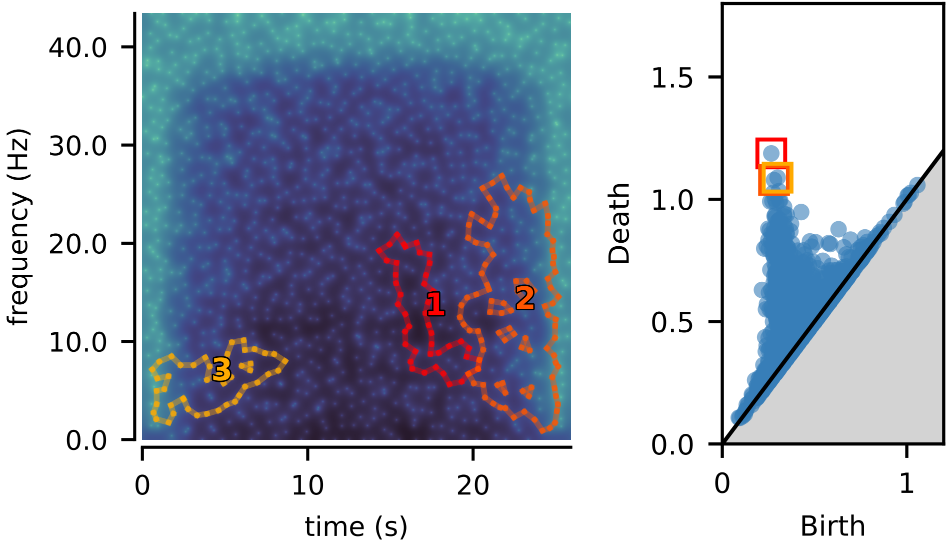


A gravitational wave



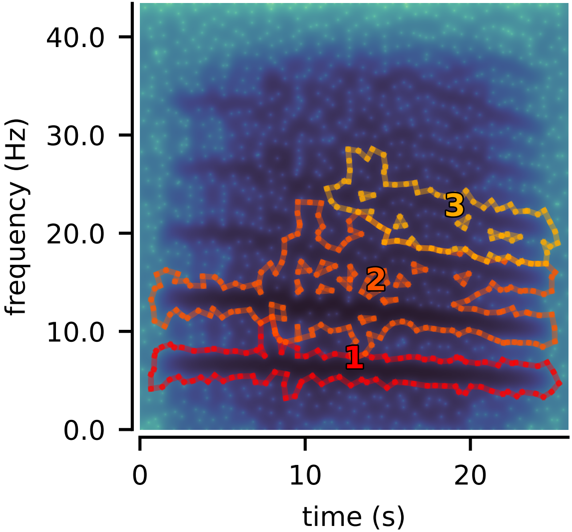


EEG data



40.0
30.0
20.0
10.0
0.0
frequency (Hz)
0
10
20
time (s)
1
2
3
1.5
1.0
0.5
0.0
Death
0
1
Birth

EEG data



40.0
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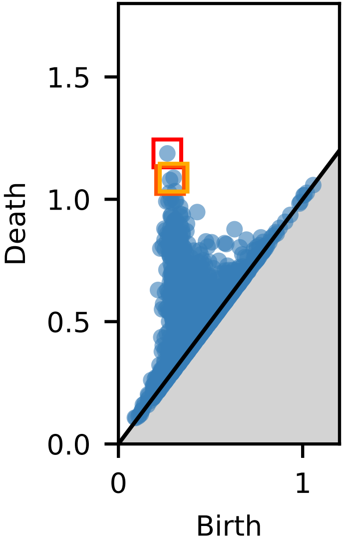
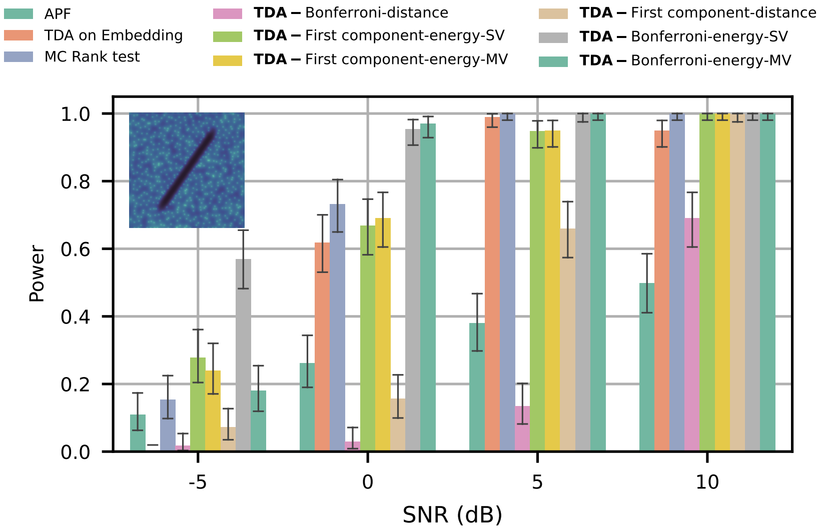
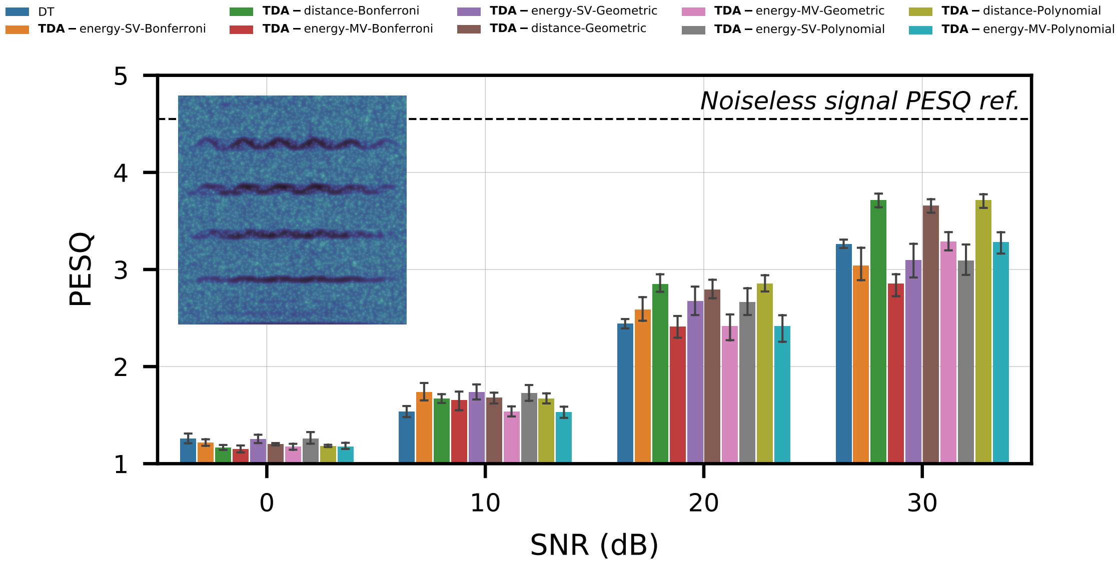


1.5
1.0
0.5
0.0
Death
0
1
Birth

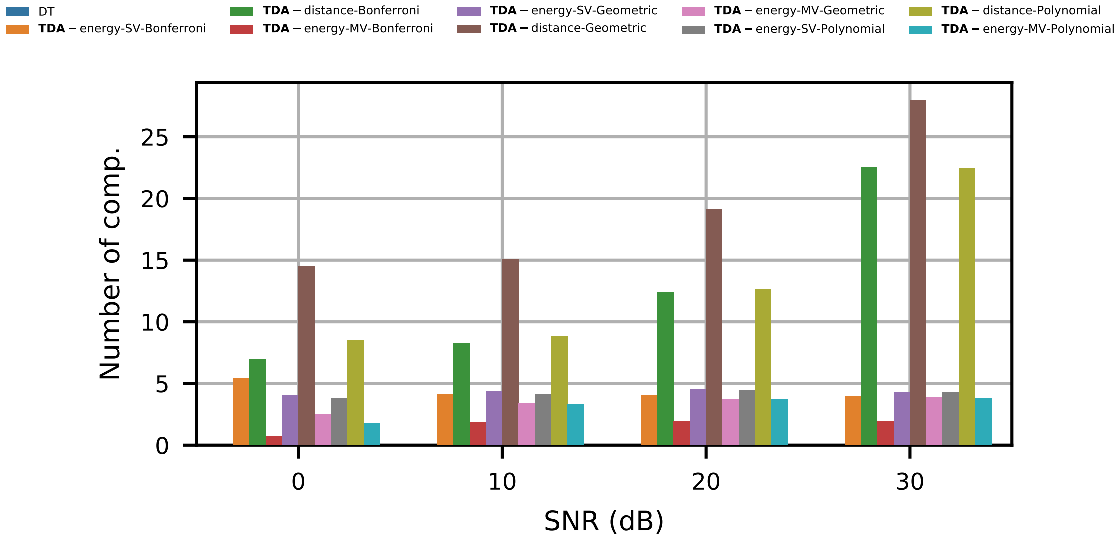
A synthetic chirp



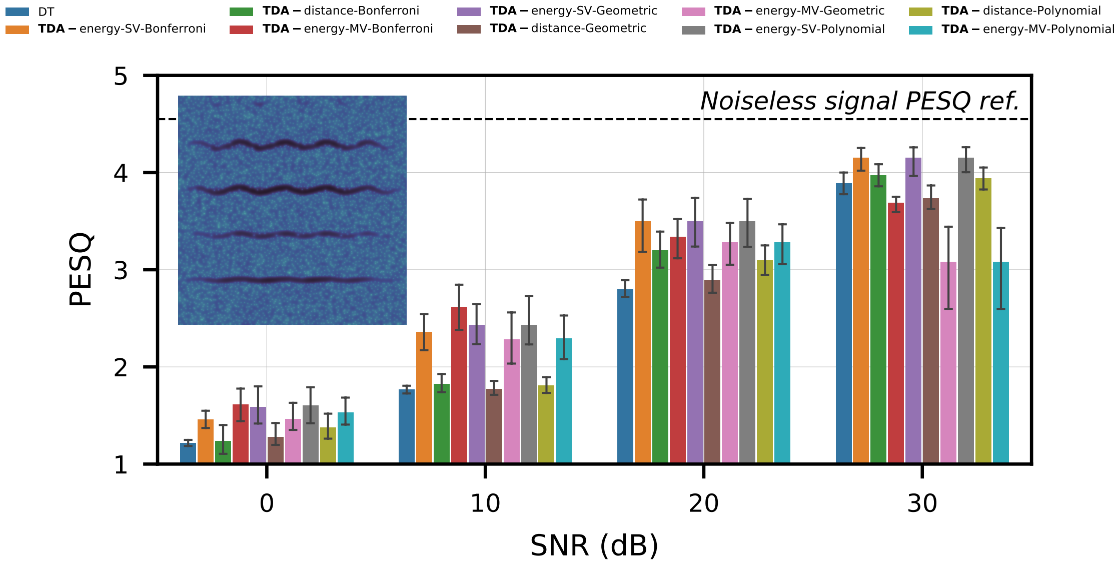
A cello recording



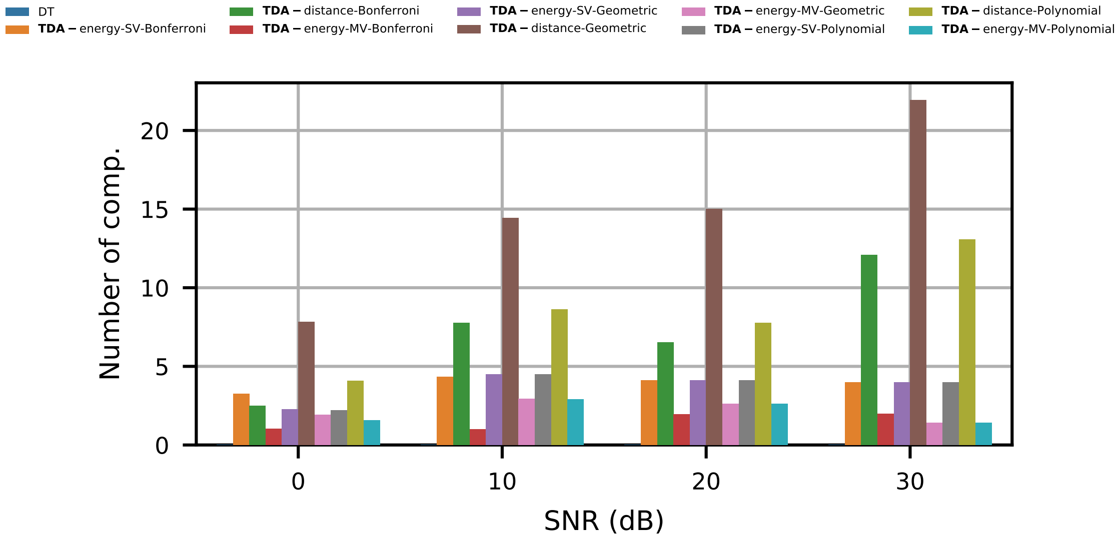
A cello recording



A violin recording



A violin recording



Conclusions

Take-home message

TDA applied to the zeros of spectrograms helps detect and reconstruct signals.

- Point pattern and holes are hidden in the time-frequency representation.
- At low SNR, there is a an advantage in using stable volumes AND the energy they contain.
- Looking into many (30!) points in the PD, with a simple Bonferroni correction, *uniformly* helps (detection vs reconstruction, MSE or PESQ)
- Estimating the number of components looks promising.

Join us for France-Morocco tonight at 10 PM!

References

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