Hyperuniform number variance bounds for Coulomb gases and Girko ensembles

Hyperuniformity and Topological Data Analysis
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Ginibre ensemble

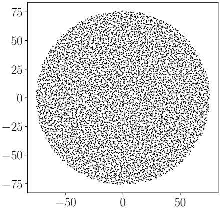


Figure: Ginibre eigenvalues

- Random **Ginibre matrix** $M = (G_{i,j})$ where the $G_{i,j}$ are i.i.d. with law $\mathcal{N}_{\mathbb{C}}(0,1)$ (i.e. real and imaginary parts are i.i.d. $\mathcal{N}(0,1)$)
- *Left:* Let $\mathsf{X}_N = \text{eigenvalues}(M) \subset \mathbb{C}$. Then for $R \leqslant \sqrt{n}$,

$$\mathbf{E}\left[\#\mathsf{X}_N \cap B(0,R)\right] \asymp R^2, \qquad \text{Var}\left(\#\mathsf{X}_N \cap B(0,R)\right) \asymp R.$$

[Ginibre '65]

Definition of hyperuniformity on scales $[1, N^a]$

- Let $\mathsf{X}_N = \{x_1, \dots, x_N\} \subset \Sigma := N^{1/d}\Sigma_0$
- **Hyperuniformity of $\mathsf{X}_N, N \to \infty$ on scales** $[1, N^a]$: for $R \in [1, N^a]$

$$\sup_{x \in \Sigma} \mathrm{Var}\left(\mathsf{X}_N(B(x,R))\right) \ll CR^d \qquad (\approx CR^{d-\alpha} \text{ for some } \alpha > 0)$$

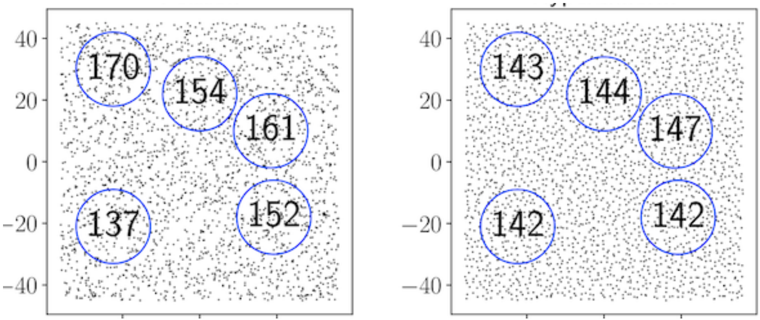


Figure: *Left:* Poisson / i.i.d. points. *Right:* "Hyperuniform disordered"

Example 1: Girko matrices (microscopic zoom)

Consider the eigenvalues $\mathsf{X}_N := \{x_1, \ldots, x_N\}$ of a $N \times N$ random matrix M with i.i.d. centred entries $M_{i,j} \in \mathbb{C}, 1 \leqslant i,j \leqslant N$ with a fixed law with all moments finite and $\mathbf{E}|M_{i,j}|^2 = 1$, e.g. **complex Ginibre ensemble:**

- **Asymptotic homogeneity:** for $f \in \mathscr{C}_c^\infty(B(0,1)), R \in [1, N^{1/2}]$

$$\mathsf{X}_N(f_R) := \sum_{i=1}^N f(x_i/R) \text{ has expectation } \sim R^2 \frac{1}{\pi} \int f$$

- **[Cipolloni, Erdös, Schröder '23]** for $f \in \mathscr{C}_c^1(B(0, 1-\varepsilon))$, "hyperuniform / Ginibre-like" universality behaviour:

$$\sup_{N \geqslant 1, R \in [1, \sqrt{N}]} \mathrm{Var}\,(\mathsf{X}_N(f_R)) \leqslant C_f < \infty, \qquad f_R(x) := f(x/R).$$

- **For comparison:** If the x_i were i.i.d. / Poisson in $B(0, \sqrt{N})$

$$\mathrm{Var}(\mathsf{X}_N(f_R)) \asymp \mathbf{E}\mathsf{X}_N(f_R) \asymp R^2$$

Girko matrices (Cont'd)

- **[Cipolloni, Erdös, Schroder '23]** hence proved hyperuniformity for smooth linear statistics. *"Extending this study to less regular f creates substantial difficulties"*.
- **[Cipolloni, Erdös, Kolupaiev '26]** For $f = 1_\Omega$ for $\Omega \subset B(0,1)$ sufficiently smooth, with some additional ("removable") assumption on the $X_{i,j}, R = N^a$ with $a \in (0, 1/2)$,

$$\mathrm{Var}\left(\mathsf{X}_N(f_R)\right) \leqslant C_a R^{2-\varepsilon}$$

where $0 < \varepsilon \leqslant 1/40$.

$\Rightarrow$ Negligible compared to the Poisson case, *hyperuniform*!

- "Expected" rate: R^1 (like for Ginibre matrices, *class I hyperuniformity*).

Example 2. Coulomb gases /OCPs (One Component Plasmas)

- Let $\mathsf{X}_N^\beta = \{x_1, \dots, x_N\} \subset \mathbb{R}^2$ with joint density

$$Z_N^{-1} \exp(-\beta \underbrace{\sum_{i \neq j} (-\ln(\|x_i - x_j\|))}_{\text{Energy}}) \times \{\text{confining term}\}$$

- **Ginibre:** case $\beta = 2$ with confinement $\prod_i \exp(-\|x_i\|^2)$
- **[Leblé '23]** Proves hyperuniformity in the bulk for all $\beta > 0, R > R_0$:

$$\sup_{x: B(x,R) \subset \sqrt{N}\Sigma_0} \mathrm{Var}\left(\mathsf{X}_N^\beta(B(x,R))\right) \leqslant C_\beta \frac{R^2}{\ln(R)^{0.6}}, \qquad \Sigma_0 = B(0, \rho_2 - \varepsilon$$

- **Poisson** (with the same scaling): $\mathrm{Var}\left(\mathsf{X}_N(B_R)\right) \asymp R^2$
- **What is expected:** $\asymp R^1$, *class I hyperuniformity*

Coulomb gases /OCPs (Cont'd)

- In dimension $d \geqslant 3$, the β-OCP has density

$$Z_N^{-1} \exp(-\beta \underbrace{\sum_{i \neq j} c_d \|x_i - x_j\|^{2-d}}_{\text{Energy}}) \times \{\text{confining term}\}$$

- **[Serfaty '23]** shows variance reduction for smooth linear statistics f

$$\mathrm{Var}\left(\mathsf{X}_N^{\beta}(f_R)\right) \leqslant C_{f,\beta} R^{2d-4}$$

for $d = 2, 3$ (3D Coulomb potential: $\| \cdot \|^{-1}$ instead of $-\ln(\| \cdot \|)$

$$d = 2 : \mathrm{Var}\left(\mathsf{P}_n(f_R)\right) \leqslant C_{f,\beta} = C_{f,\beta} R^{d-2} \quad \text{“2-hyperuniformity”}$$

$$d = 3 : \mathrm{Var}\left(\mathsf{P}_n(f_R)\right) \leqslant C_{f,\beta} R^2 = C_{f,\beta} R^{d-1} \quad \text{“1-hyperuniformity”}$$

- *"Treating the case of nonsmooth kernels (...) remains a (significantly more) delicate problem."*
- **[Chatterjee '17]** Proves 2-HU for the $3D$ hierarchical Coulomb gas

Variance transfer principle

Goal of this work: prove that if the rate for smooth linear statistics is in $R^{d-\alpha}$, then the rate for balls is $R^{d-\min(\alpha,1)}$ ($\times \ln(R)$ if $\alpha = 1$).

- $\alpha > 0$ is the *hyperuniformity* exponent
- Girko eigenvalues/Coulomb gases: α is conjectured to be 2
- Hence we should obtain "surface order growth", i.e. class I hyperuniformity

$$\mathrm{Var}\left(\mathsf{X}_N(B(0,R))\right) \sim R^{d-1}$$

- **[Beck '87]** This is the minimal possible rate for a "homogeneous system of points"

Original definition for infinite processes

- $\mathcal{N}(\mathbb{R}^d)$: **Space of configurations:** locally finite sets, endowed with vague topology (local). A point configuration is seen as a measure

$$\mathsf{P}(f) = \sum_{x \in \mathsf{P}} f(x).$$

- **Point process** P : Random element of $\mathcal{N}(\mathbb{R}^d)$
- **Stationarity:** invariance in law under translations $\tau_x : y \mapsto x + y$

Definition ([Torquato, Stillinger, '03])

A stationary point process P is hyperuniform (HU) if

$$\mathrm{Var}\,(\mathsf{P}(B_R)) = o(R^d) \text{ where } B_R := B(0, R)$$

- For P a stationary Poisson process, $\mathrm{Var}\,(\mathsf{P}(B_R)) = \lambda R^d$ (not HU)

Isomorphism theorem and spectral characterisation

- Let P be a stationary point process with $\mathbf{E}\left[\mathsf{P}(B(0,1))^2\right] < \infty$.
- There is a tempered measure S_P (the "spectral measure") giving a Plancherel type formula

$$\mathrm{Var}\left(\mathsf{P}(f)\right) = \int |\hat{f}|^2 d\mathsf{S}_P, f \in \mathscr{B}_c^{\mathsf{b}}(\mathbb{R}^d) \quad \text{(bounded with compact support)}$$

Hyperuniformity is a global variance cancellation phenomenon over linear statistics

Theorem (Torquato, Björklund & Hartnick)

The following are equivalent

1. $Var(\mathsf{P}(B_R)) = o(R^d), R \to \infty$
2. $\mathsf{S}_{\mathsf{P}}(B(0,\varepsilon)) = o(\varepsilon^d), \varepsilon \to 0$
3. $Var(\mathsf{P}(f_R)) = o(R^d), R > 1$ *for some* $f \in \mathscr{B}_c(\mathbb{R}^d)$ *such that* $\int f \neq 0$

Hyperuniformity exponent

- **Definition:** P is α-hyperuniform if

$$\mathsf{S}_{\mathsf{P}}(B(0,\varepsilon)) = O(\varepsilon^{d+\alpha})$$

Classical case: S_{P} has a density of class $\mathscr{C}^2 \Rightarrow$ 2-hyperuniform

Proposition

The three following are equivalent

(i) P *is* α*-hyperuniform*

(ii) $\exists f \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ such that $\int f \neq 0$, $Var(\mathsf{P}(f_R)) = O(R^{d-\alpha})$.

(iii) *For all* f *sufficiently regular,* $Var(\mathsf{P}(f_R)) = O(R^{d-\alpha})$

A bounded compactly supported function f is α-regular if for some $\gamma > \alpha$

$$|\hat{f}(\xi)|^2 = O(\|\xi\|^{-d-\gamma})$$

1D / 2D proven examples

- $d = 1$: β-ensembles / Sine_β / $\log$-gases, with $\alpha = 1$ (optimal)
- $d = 2$: **Infinite Ginibre ensemble**, stationary and hyperuniform with $\alpha = 2$ (optimal)
- **[Sodin & Tsirelson '06]** Let $F(z)$ be the planar GAF (Gaussian Analytic Function, limit of *Weyl polynomials*)

$$F(z) = \sum_{k=0}^{\infty} \frac{X_k z^k}{\sqrt{k!}}, \; X_k \sim \mathcal{N}_{\mathbb{C}}(0,1), \text{ i.i.d.}$$

  and its zeros $\mathscr{Z} = \{z : F(z) = 0\}$. Then $\mathscr{Z}$ is stationary and hyperuniform with $\alpha = 4$, this is the unique "non-toy model" with (proven) $\alpha > 2$.
- **[Joyce et al. '08, Lr '24, Lotz & Klatt '26]** Toy models p-hyperuniform for $p \to \infty$

Proof of the transfer principle and generalisation

$\varphi = 1_{B(0,1)}$ is 1-regular:

$$|\hat{\varphi}(\xi)|^2 \leqslant C(1+\|\xi\|)^{-d-1}$$

- The proof is based on high / low frequency decomposition, using **translation boundedness** of the spectral measure.
- Assume $\mathsf{S}(d\xi) \sim \|\xi\|^\alpha$ as $\xi \to 0$ for simplification

$$\begin{aligned}\mathrm{Var}\,(\mathsf{P}(\varphi_R)) &= \int_{\mathbb{R}^d} |\widehat{\varphi_R}|^2 d\mathsf{S} = R^{2d}\int_{\mathbb{R}^d} |\hat{\varphi}(R\xi)|^2 \mathsf{S}(d\xi)\\ \approx & R^{2d}\int_{B(0,\varepsilon/R)} |\hat{\varphi}(0)|^2 \|\xi\|^\alpha d\xi \\ &\qquad + R^{2d}\int_{B(0,\varepsilon/R)^c} (1+\|R\xi\|)^{-d-1}(``\|\xi\|^\alpha \wedge O(1)")d\xi \\ \leqslant & (....) \leqslant CR^{d-\min(\alpha,1)}\ln(R)^{\mathbf{1}\{\alpha=1\}}.\end{aligned}$$

Similar result on the torus

- Let P_n be a point process on $\mathbb{T}_n := \mathbb{R}^d/(n\mathbb{Z})^d$
- Let τ_x^n the n-periodic translation on $\mathbb{T}_n$
- Say that P_n is $\mathbb{T}_n$-*stationarity* if:

$$\tau_x^n \mathsf{P}_n \overset{(d)}{=} \mathsf{P}_n, x \in \mathbb{T}_n.$$

- **Example:** Coulomb gas with periodic boundary conditions: $\mathsf{P}_n = \{x_1, \ldots, x_n\} \subset \mathbb{T}_n$ with density

$$\propto \exp(-\psi_d(x_1, \ldots, x_n))$$

where

$$\Delta_{\mathbb{T}_n} \psi_d = \delta_0 - n^{-d} \mathbf{Leb}^d$$

Theorem (Lr '26)

Let P_n *be a* $\mathbb{T}_n$*-stationary point process with unit intensity.*
Assume P_n *is* α*-hyperuniform for some smooth kernel* $f \in \mathscr{C}_c^\infty(\mathbb{R}^d) \setminus \{0\}$*:*
for $R \in [1, n^\gamma]$ *(*$\gamma \in [0,1]$*)*

$$Var(\mathsf{P}_n(f_R)) \leqslant C_f R^{d-\alpha}$$

then for $R \in [1, n^\gamma]$

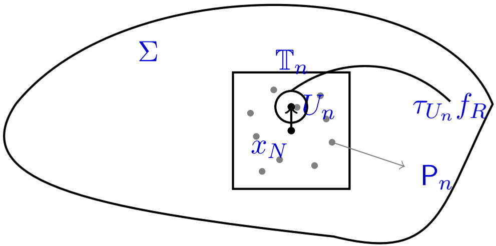
$$\sup_x Var(\mathsf{P}_n(B(x,R))) \leqslant C_f R^{d-\min(\alpha,1)} \ln(R)^{\mathbf{1}\{\alpha=1\}}$$

The proof is similar to the $\mathbb{R}^d$ case:

- The Fourier dual space of $\mathbb{T}_n$ is $\widehat{\mathbb{T}_n} := 2\pi n^{-1}\mathbb{Z}^d$
- There is a similar isomorphism theorem with a spectral measure S_n on $\widehat{\mathbb{T}_n}$. In some sense, $\widehat{\mathbb{T}_n} \xrightarrow[n\to\infty]{} \mathbb{R}^d$
- A similar high/low frequency decomposition is possible

Adaptation to “almost stationary” samples

- X_N process on $\Sigma = B(0, N^{1/d})$ (or else)
- P_n : restriction to $x_N + \mathbb{T}_n$ with some $n \ll N^{1/d}$
- Let U_n uniform on $\mathbb{T}_n$, $\tilde{\mathsf{P}}_n := \tau^n_{U_n}\mathsf{P}_n$ stationary by construction
- Hopefully $\mathrm{Var}\left(\tilde{\mathsf{P}}_n(\tau_x f_R)\right) \approx \mathrm{Var}\left(\mathsf{P}_n(\tau_x f_R)\right)$ in law, unfortunately boundary effects kill some available scales



Consequences for infinite Coulomb gases / OCPs

Let X^β be an *infinite volume Coulomb gas*, i.e.

$$\mathsf{X}^\beta = \lim_{N_j \to \infty} \mathsf{X}^\beta_{N_j}$$

is a "limit point in law" on $\mathbb{R}^d$ for the vague topology. The previous method yields the following:

Theorem

- *In dimension $d = 2$, any limit point P is stationary and P 2-hyperuniform, i.e. for all $x \in \mathbb{R}^d$*

$$Var\left(\mathsf{X}^\beta(B(x,R))\right) \leqslant C_\beta R < \infty.$$

- *In dimension $d = 3$, it gives 1-hyperuniformity if P is assumed to be stationary.*

Perturbed lattices in dimension $d = 2$

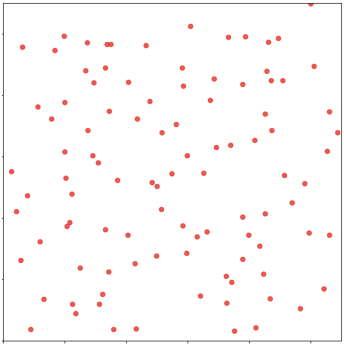
This proves **finite Coulomb energy** and **finite Wasserstein distance** to Lebesgue measure by **[Lr & Yogeswharan '24, Huesmann & Leblé '25]**. It means X^β is a L^2-perturbed lattice.

- Red: Coulomb gas / OCP particles of X^β
- Blue: Lattice points $k \in \mathbb{Z}^d$
- Gray arrows: Vectors $U_{\mathbf{k}}$ such that

$$\mathsf{X}^\beta = \{k + U_{\mathbf{k}}; k \in \mathbb{Z}^d\}$$

$$\{U_{\mathbf{k}}\} \stackrel{(d)}{=} \{U_{\mathbf{k}+\mathbf{k}_0}\}, \mathbf{k}_0 \in \mathbb{Z}^d$$

- L^2-perturbed lattice $\Leftrightarrow$ $\mathbf{E}\|U_0\|^2 < \infty$.



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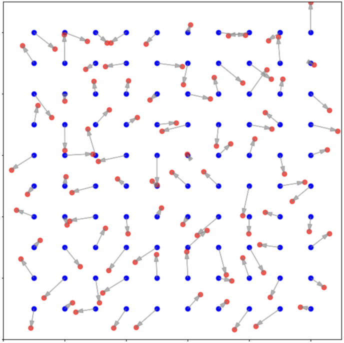


- L^2-perturbed lattice $\Leftrightarrow$ $\mathbf{E}\|U_0\|^2 < \infty$.

Non-asymptotic results

Building on results of **[Serfaty '23]** about smooth linear statistics

Theorem (Lr '26)

Let $R \in [1, N^a]$ *with* $a(d+2) < 1/d$, *then for some* $0 < \delta < 1/d$,

- $d = 2, \beta > 0$: *for* $x \in B(0, (\rho_2 - \varepsilon)\sqrt{N})$,

$$\frac{1}{(N^\delta)^d} \int_{\mathbb{T}_{N^\delta}} Var\left(\mathsf{X}_N^\beta(B(x,R))\right) \leqslant C_\beta R \qquad \ll R^2$$

- $d = 3, \beta > 0$: *for* $x \in B(0, (\rho_3 - \varepsilon)N^{1/3})$,

$$\frac{1}{(N^\delta)^d} \int_{\mathbb{T}_{N^\delta}} Var\left(\mathsf{X}_N^\beta(B(x,R))\right) \leqslant C_\beta R^2 \ln(R) \qquad \ll R^3.$$

It prevents the rate from being $\gg R^{d-1}\ln(R)$ outside a vanishing set.

Results for Girko random matrices

Building on the results of **[Cipolloni et al. '23]** about smooth linear statistics, there is number variance hyperuniformity "almost everywhere" (but not at all scales)

Theorem (Lr '26)

Let $R \in [1, N^{\frac{1}{8}}], \delta < 1/2 - 1/8, x \in B(0, (1-\varepsilon)\sqrt{N})$

$$\frac{1}{N^{2\delta}} \int_{\mathbb{T}_{N^\delta}} Var(\mathsf{X}_N(B(x,R)))\, dx \leqslant CR^1$$

Comparison with [Cipolloni et al. '26]

- Scales $[1, N^{1/8}]$
- No uniform result in x
- But optimal rate: R^1 instead of $R^{2-\varepsilon}$.

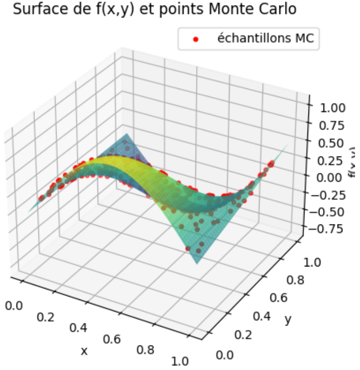
Thank you very much!

BONUS TRACK

Numerical integration / Monte Carlo

Let f be a "complicated" function on a "high-dimensional" cube of dimension d. What is $\int f$?
Let $x_1, \dots, x_n$ be random points.

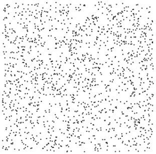
$$\int f \sim \frac{1}{n} \sum_{i=1}^{n} f(x_i)?$$



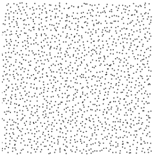
If the x_i are independent and identically distributed ("i.i.d."),

$$\left| \int_{[0,1]^d} f - \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right| \sim n^{-1/2}$$

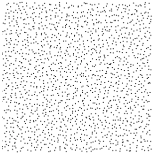
Monte Carlo integration with hyperuniform samples



i.i.d. sample: Error $\sim n^{-1/2}$

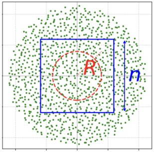


Ginibre eigenvalues: Error $\sim n^{-1}$



Weyl zeros: Error $\sim n^{-3/2}$

Finite systems and "Bulk"



Surprisingly, a similar phenomenon occurs for large finite systems (random matrices, Coulomb / Riesz gases, ...)

Theorem (Lr '26)

Let P_n *be a random configuration on* $\mathbb{T}_n^d = [0,n]^d$ *invariant under the action of translations (i.e.* $\mathsf{P}_n \overset{(d)}{=} \tau_x^{\mathbb{T}_n^d}\mathsf{P}_n$*). If* $\exists \alpha \in \mathbb{R}, f \in \mathscr{C}_c^\infty(\mathbb{T}_1^d)$ *such that*

- $\forall R \in [1,n], \mathit{Var}(\mathsf{P}_n(f(\cdot/R))) \leqslant cR^{d-\alpha}$

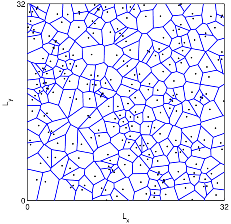
then

- $\forall R \in [1,n], \underbrace{\varepsilon R^{d-1} \leqslant}_{\text{[Beck '87]}} \mathit{Var}(\mathsf{P}_n(B_R)) \leqslant c' R^{d-\min(\alpha,1)} \ln(R)^{\mathbf{1}\{\alpha=1\}}$

There are additional corrective terms if the translation invariance is only

Optimal transport

Balanced partitions



The typical cell C of a Poisson-Voronoi tessellation behaves well:

$$\mathbf{P}(\mathrm{diam}(C) > r) \leqslant Ce^{-\kappa r^d}.$$

What if the tessellation has to be *balanced*, i.e. cells must have volume 1?
It is an optimal transport problem: for $P = \{x_1, \dots, x_{n^d}\}$,
$\mathbf{Leb}_n^d = \mathbf{Leb}^d 1_{\mathbb{T}_n^d}$

$$\mathsf{W}_p^p(P, \mathbf{Leb}_n^d) := \inf_{\{C_i\} \text{ fair partition}} \sum_{i=1}^{n^d} \int_{C_i} \|x_i - x\|^p dx$$

Remark: Points can fall outside the cells

AKT theorem

For i.i.d. points, the average cell behaves badly in dimensions $1, 2$:

Theorem (Ajtai, Komlós, Tusnády)

Let $X_1, \dots, X_{n^d}$ *i.i.d. uniform in* $\mathbb{T}_n^d$

$$\lim_n \frac{\mathbf{E}\left[\mathrm{W}_p^p(\{X_1, \dots, X_{n^d}\}, \mathbf{Leb}_n^d)\right]}{n^d} < \infty \text{ iff } p < d/2 \text{ or } d \geqslant 3.$$

What about hyperuniform processes?

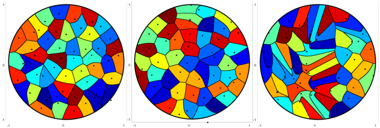


Figure: **[Jalowy '23]** Hyperuniform or not?

HU and optimal transport

Specific models: **[Chafai et al. '18, Prod-homme '21, Jalowy '23]**

Theorem ([Lr, Yogeshwaran '24], [Huessman, Leblé '24])

Let P *be a stationary point process such that*

$$\int_1^\infty R^{-2d+1}\,Var(\mathsf{P}(B_R))\,dR < \infty \qquad \text{(HU +)}$$

- $\mathsf{P}_n := \mathsf{P} \cap \mathbb{T}_n^d$
- $\tilde{\mathsf{P}}_n := \frac{n^d}{\#\mathsf{P}_n}\mathsf{P}_n$

Then

$$\mathsf{W}_2^2(\tilde{\mathsf{P}}_n, \mathbf{Leb}_n^d) \leqslant cn^d$$

- $d \geqslant 3$: satisfied by Poisson ($\mathsf{P}_n \approx n^d$ points i.i.d)
- $d = 2$: requires hyperuniformity (e.g. $\mathrm{Var}\,(\mathsf{P}(B_R)) = O(R^2\ln(R)^{-1-\varepsilon})$)

Reverse Sobolev

- Let $\mathsf{S}_{\mathsf{P}_n}$ the spectral measure P_n, i.e. for $f \in \mathscr{C}_c^\infty(\mathbb{T}_n^d)$

$$\mathsf{S}_{\mathsf{P}_n}\left(\frac{k}{n}\right) = n^{-d}\mathbf{E}\left|\mathsf{P}_n\left(e^{i\langle\cdot,k/n\rangle}\right)\right|^2, \frac{k}{n} \in \mathbb{Z}_n^d$$

  (Random matrices: "The spectrum's spectrum"...)
- Let $f_n\mathbf{Leb}_n^d = \tilde{\mathsf{P}}_n *_{\mathbb{T}_n^d} \psi$ be a regularisation of P_n
- Reverse Sobolev inequality

$$\mathsf{W}_2^2(f_n\mathbf{Leb}_n^d, \mathbf{Leb}_n^d) \leqslant \int_{\mathbb{T}_n^d} |\nabla\Delta^{-1}(f_n - 1)|^2 = c_{d,n} \sum_{k\in\mathbb{Z}_n^d\setminus\{0\}} \frac{|\widehat{f_n}(k)|^2}{\|k\|^2}$$

  and $\mathbf{E}|\widehat{f_n}|^2 = \mathbf{E}|\widehat{\tilde{\mathsf{P}}_n}|^2|\hat{\psi}|^2 \sim \mathsf{S}_{\mathsf{P}_n}$
- The right hand member converges to $c_d \int_{B(0,1)} \frac{\mathsf{S}_{\mathsf{P}}(du)}{\|u\|^2}$

Coulomb energy

- Fourier computations

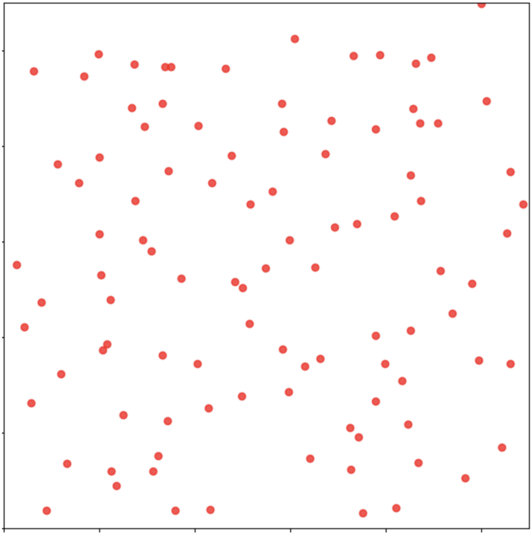
$$\int_{B(0,1)} \frac{\mathsf{S}_{\mathsf{P}}(du)}{\|u\|^2} < \infty \;\Leftrightarrow\; \int \mathrm{Var}\left(\mathsf{P}(B_R)\right) R^{-2d+1} dR < \infty$$

- **[Huesmann, Leblé '23]** show that $\int_{B(0,1)} \frac{\mathsf{S}_{\mathsf{P}}(du)}{\|u\|^2} < \infty$ is equivalent to a *finite Coulomb energy* in dimension $d = 2$
- What is Coulomb energy? Roughly speaking

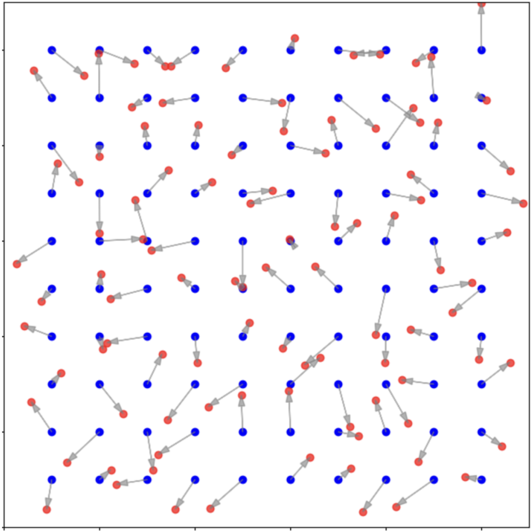
$$\lim_n \int_{x \neq y} \ln(\|x-y\|)(\mathsf{P}_n - \mathbf{Leb}^d)(dx)(\mathsf{P}_n - \mathbf{Leb}^d)(dy)$$

- Rigourously defined as the existence of an electric field $\mathrm{Coul}_{\mathsf{P}}$ such that $\mathrm{div}(\mathrm{Coul}_{\mathsf{P}}) = \mathsf{P}_n - \mathbf{Leb}^d$ (**[Serfaty, Sodin, ...]**)

Perturbed lattices



Perturbed lattices



Perturbed lattices

- We have a triangle inequality

$$W_2^2(\tilde{\mathsf{P}}_n, \mathbb{Z}_n^d) \leqslant \mathsf{W}_2^2(\tilde{\mathsf{P}}_n, \mathbf{Leb}_n^d) + \mathsf{W}_2^2(\mathbf{Leb}_n^d, \mathbb{Z}_n^d) \leqslant cn^d$$

  implies the existence of a transport $T_n : \mathbb{Z}_n^d \to \tilde{\mathsf{P}}_n$

- Through a compactness argument, there is a bijection $T : \mathbb{Z}^d \to \mathsf{P}$, $\mathbb{Z}^d-$stationary such that

$$W_2^2(\mathbb{Z}^d, \mathsf{P}) := \mathbf{E}\|T(0)\|^2 = \mathbf{E}\|T(k)\|^d < \infty.$$

- We say that P is a L^2-perturbed lattice:

$$\mathsf{P} = \{k + T(k); k \in \mathbb{Z}^d\}.$$

Rigidity

Rigidity

- **[Ghosh, Peres '17]** put in evidence the phenomenon of **Number rigidity** (NR): for $\mathsf{P} = \mathsf{P}^{\mathrm{Gin}}$ or $\mathsf{P} = \mathsf{P}^{\mathrm{GAF}}$,

$$\#\mathsf{P} \cap B_1 \in \sigma(\mathsf{P} \cap B_1^c)$$

- Bufetov and co-authors proved similar results for DPPs and gave a sufficient condition for $d = 2$

[Ghosh, Lebowitz '17] give sufficient conditions in dimensions 1 et 2 if S has a density s:

Theorem ([Ghosh, Lebowitz '17])

- *Dimension* 1*:* **NR** *si* $\mathsf{s}(u) \leqslant c|u|$ *as* $u \to 0$ *(1-hyperuniformity)*
- *Dimension* 2*:* **NR** *si* $\mathsf{s}(u) \leqslant c\|u\|^2$ *as* $u \to 0$ *(2-hyperuniformity)*

Biblio

- **[Aizenman '80]** First example (Coulomb gas)
- **[Dereudre et al. '20, Najnudel et '18, Thoma '23, Chatterjee '19]** Rigidity of Coulomb gases
- **[Bufetov et co-auteurs]** DPPs, Pfaffian processes
- **[Bjorklund, Hartnick '23]** Quasicrystals
- Gaussian zeros, eigenvalues of random operators, ...
- Those rigidities are all linear: in $L^2(\mathbf{P})$

$$\#\mathsf{P} \cap B_1 = \lim_n \mathsf{P}(h_n) \text{ where } h_n \in \mathscr{C}_c^\infty(B_1^c)$$

Linear k-rigidity

- **[Ghosh, Peres '17]** The moment of order 1 of $\mathsf{P}^{\mathrm{GAF}}$ is also linearly approximable

$$\sum_{x\in\mathsf{P}^{\mathrm{GAF}}} x = \lim_n \mathsf{P}(h_n) \text{ with } h_n \in \mathscr{C}_c^\infty(B_1^c)$$

but not more!

- Define linear k-rigidity as the $L^2(\mathbf{P})-$approximability

$$\int_{B_1} x_1^{k_1}\dots x_d^{k_d} d\mathsf{P}(x) = \lim_n \mathsf{P}(h_n) \text{ with } h_n \in \mathscr{C}_c^\infty(B(0,1)^c), \sum_i k_i = k$$

- **NR** $=$ 0-rigidity
- Linear rigidity is a Hilbert space approximation problem

$$0 \stackrel{?}{=} \inf_{h\in\mathscr{C}_c^\infty(B_1^c)} \mathrm{Var}\left(\mathsf{P}(1_{B_1}) - \mathsf{P}(h)\right) = \inf_h \mathsf{S}_{\mathsf{P}}(|\widehat{1_{B_1}} - \hat{h}|^2)$$

Applications

- A standard "disordered" hyperuniform process P satisfies $\mathsf{S}(du) = \mathsf{s}(u)du$ with $\mathsf{s}(u) \geqslant c\|u\|^2$. One has k-rigidity iff

$$\int_{B(0,1)} \frac{\|u\|^{2k}}{\|u\|^2} du = \infty \ \Leftrightarrow \ k = 0, d = 1, 2.$$

Theorem (Lr '24)

Let P be a stationary DPP with reduced kernel $\kappa \in L^2(\mathbb{R}^d)$. Then P is **NR** *iff $d = 1, 2$, and*

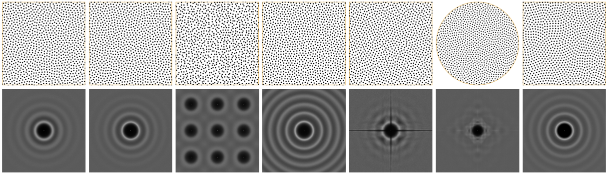
$$\int \frac{1}{1 - \widehat{\kappa^2}(u)} du = \infty,$$

and never k-rigid for $k > 1$.

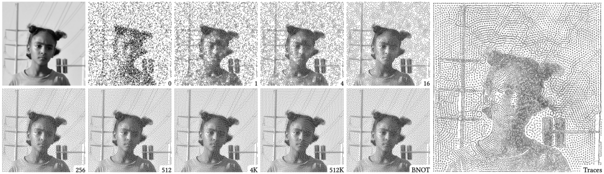
"Stealthy" process

Extreme hyperuniformity: stealthy processes

- A stationary process P is *stealthy* if it has a spectral gap: $\exists\varepsilon > 0$ such that $\mathsf{S}_{\mathsf{P}}(B(0,\varepsilon)) > 0$ (hence it is ∞-HU / ∞-rigid)
- There are many practical works in physics / signal processing to generate such samples

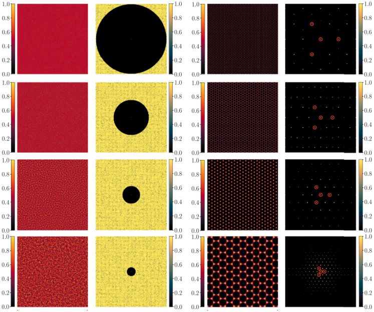


(a) BNOT (b) KDM (c) VnC (d) FPO (e) BlueNets (f) SOT (g) GBN (Ours)



Transition de phase [Torquato et al.]

If one maintains the intensity of points, there is a phase transition as ε grows:



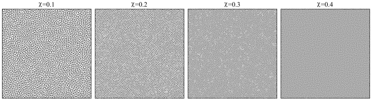


FIG. 1. Stealthy hyperuniform point patterns of $N = 2 \times 10^6$ particles with $\chi = 0.1, 0.2, 0.3$, and 0.4. Each image only shows 1/16th of all the data for better visualizations. Full configuration data are deposited through Princeton Data Commons [35].

Theoretical questions

- The only rigourously proven stealthy processes are shifted / rotated / "slided" independent copies of $\mathbb{Z}^d$
- **Are there infinite processes** P **stealthy and "disordered"?** (mixing / with non-atomic spectrum)
- Do such processes have a W_∞-distance uniformly bounded?