Rate of convergence of empirical measures of hyperuniform point processes

Sandrine Dallaporta (Université de Poitiers)

Hyperuniformity and Topological Data Analysis - July 9th 2026

A very classical question

Consider N i.i.d. points $(X_1, \ldots, X_N)$ of law μ on $\mathbb{R}^d$.

Empirical measure $\mu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k}$.

$$\text{LLN:} \quad \mu_N \xrightarrow[N\to\infty]{\text{a.s.}} \mu.$$

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Case $d = 1$: F_N and F be the respective c.d.f's of μ_N and μ.

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More precisely, $\sqrt{N}\|F_N - F\|_\infty \xrightarrow[N\to\infty]{\text{(d)}} \text{KS}$.

$$\|F_N - F\|_\infty \approx O\left(\frac{1}{\sqrt{N}}\right).$$

Wasserstein distances

Definition

Let $p \geqslant 1$. If μ and ν are two positive measures with a finite p-th moment s.t. $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$, define

$$W_p(\mu, \nu) = \left(\inf_{\Pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p d\Pi(x, y) \right)^{1/p},$$

where the infimum is taken on all positive measures Π on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν.

W_p convergence $\Leftrightarrow$ weak convergence $+$ moment convergence

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W_p convergence $\Leftrightarrow$ weak convergence $+$ moment convergence

Link between these distances:

- $W_1(\mu,\nu) \leqslant \mu(\mathbb{R}^d)^{1-\frac{1}{p}} W_p(\mu,\nu)$.
- $\|F_\mu - F_\nu\|_\infty \leqslant C\sqrt{W_1(\mu,\nu)}$ if ν has a bounded density w.r.t. Lebesgue.

The case of the uniform distribution: $d\mu = \mathbb{1}_{[0,1]^d} d\mathrm{Leb}$

Theorem (Ajtai-Komlós-Tusnady (1984), Talagrand (1994), Ledoux (2019))

$$\mathbb{E}[W_p^p(\mu_N, \mu)]^{1/p} \leqslant C_{d,p} \times \begin{cases} N^{-1/2} & \text{if } d = 1 \\ N^{-1/2}\sqrt{\log N} & \text{if } d = 2 \\ N^{-1/d} & \text{if } d \geqslant 3, \end{cases}$$

and there exist corresponding lower bounds.

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Theorem ($d = 2$, Ambrosio-Stra-Trevisan (2019))

If μ is the uniform distribution on $[0,1]^2$, then

$$\mathbb{E}[W_2^2(\mu_N, \mu)]^{1/2} \sim \frac{1}{2\sqrt{\pi}}\sqrt{\frac{\log N}{N}}.$$

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Fournier-Guillin (2015): optimal inequalities for more general μ.

Two usual scalings

- **empirical measure:**

$$\mu_N = \frac{1}{N}\sum_{k=1}^{N}\delta_{X_k},$$

with i.i.d. X_k's of law $d\mu = \mathbb{1}_{[0,1]^d}d\mathrm{Leb}$

$\rightsquigarrow$ μ_N of total mass 1

$\rightsquigarrow$ μ with fixed compact support (of volume 1)

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$\rightsquigarrow$ μ_N of total mass 1

$\rightsquigarrow$ μ with fixed compact support (of volume 1)

- **counting measure:**

$$L_N = \sum_{k=1}^{N} \delta_{Y_k},$$

with i.i.d. Y_k's of law $dL = \mathbb{1}_{[0,N^{1/d}]^d} d\mathrm{Leb}$

$\rightsquigarrow$ L_N of total mass N

$\rightsquigarrow$ L with "big" compact support (of volume N).

AKT result in both scalings

Theorem (scaling: empirical measure)

If μ is the uniform distribution on $[0,1]^d$, then

$$\mathbb{E}[W_p^p(\mu_N,\mu)] \leqslant C_{d,p} \times \begin{cases} N^{-p/2} & \text{if } d = 1 \\ N^{-p/2}(\log N)^{p/2} & \text{if } d = 2 \\ N^{-p/d} & \text{if } d \geqslant 3. \end{cases}$$

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Theorem (scaling: counting measure)

If L is the Lebesgue measure on $[0,N^{1/d}]^d$, then

$$\mathbb{E}[W_p^p(L_N,L)] \leqslant C_{d,p} \times \begin{cases} N^{1+p/2} & \text{if } d = 1 \\ N(\log N)^{p/2} & \text{if } d = 2 \\ N & \text{if } d \geqslant 3. \end{cases}$$

A deterministic lower bound in the continuous case

Lemma (scaling: empirical measure)

Suppose that μ is a ***probability measure*** *with bounded density with respect to Lebesgue measure on $\mathbb{R}^d$. Let $(x_1, \ldots, x_N)$ be any points in $\mathbb{R}^d$. Then*

$$W_p^p\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \mu\right) \geqslant C_{d,\mu} N^{-p/d}.$$

Remarks:

- In dimension 1 and 2, the i.i.d. uniform points rate does not match the lower bound.
- For $d \geqslant 3$, it matches the lower bound.

A deterministic lower bound in the continuous case

Lemma (scaling: counting measure)

*Suppose that L is a **measure of mass** N with bounded density with respect to Lebesgue measure on $\mathbb{R}^d$. Let $(y_1, \dots, y_N)$ be any points in $\mathbb{R}^d$. Then*

$$W_p^p\Big(\sum_{i=1}^{N} \delta_{y_i}, L\Big) \geqslant C_{d,L} N.$$

Remarks:

- In dimension 1 and 2, the i.i.d. uniform points rate does not match the lower bound.
- For $d \geqslant 3$, it matches the lower bound.

An example of non i.i.d. points in dimension 1

W_N a Wigner matrix: Hermitian or real symmetric matrix with iid entries of mean 0 and variance 1.

Important example: GUE and GOE (Gaussian entries).

ESD: $\mu_N = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_i}$, where λ_i's are the (real) eigenvalues of $\frac{1}{\sqrt{N}} W_N$.

Theorem (Wigner (1955), Pastur (1972))

$$\mu_N \xrightarrow[N\to\infty]{a.s.} \rho_{sc},$$

where

$$d\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{[-2,2]}(x)\,dx.$$

Theorem (Götze-Tikhomirov *et al* (2002-2016), Claeys-Fahs-Lambert-Webb (2021), Bourgade-Lopatto-Zeitouni (2025))

Under moment conditions,

$$\mathbb{E}[\|F_N - F_{sc}\|_\infty] \leqslant \frac{C(\log N)^2}{N}.$$

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Under moment conditions,

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Under moment conditions,

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Theorem (Guionnet-Zeitouni (2000), Meckes-Meckes (2013), D. (2012))

Under moment conditions,

$$\mathbb{E}[W_p^p(\mu_N, \rho_{sc})] \leqslant CN^{-p}(\log N)^{p/2}.$$

Recall the lower bound: $\mathbb{E}[W_p^p(\mu_N, \rho_{sc})] \geqslant cN^{-p}$.

In dimension 2

M_N a Girko matrix: a real or complex matrix with iid entries of mean 0 and variance 1.

Special case: Ginibre matrix (Gaussian entries).

ESD: $\mu_N = \frac{1}{N}\sum_{k=1}^{N} \delta_{z_i}$, where z_i's are the (complex) eigenvalues of $\frac{1}{\sqrt{N}} M_N$.

Theorem (Circular law: Mehta (1965), ..., Tao-Vu (2009))

$$\mu_N \xrightarrow[N\to\infty]{a.s.} \rho,$$

where

$$d\rho(z) = \frac{1}{\pi} \mathbb{1}_{D(0,1)}(z) dz.$$

Theorem (Meckes-Meckes (2014), Chafaï-Hardy-Maïda (2018), Prod'homme (PhD thesis 2021), Jalowy (2023))

$$\textit{(Ginibre)} \qquad \mathbb{E}[W_p^p(\mu_N,\rho)] \leqslant CN^{-p/2}, \quad p \in [1,2].$$

$$\textit{(Girko)} \qquad \mathbb{E}[W_p^p(\mu_N,\rho)] \leqslant CN^{-1/2+\varepsilon}, \quad p = 1 \text{ or } p \geqslant 2.$$

Recall the lower bound: $\mathbb{E}[W_p^p(\mu_N,\rho)] \geqslant cN^{-p/2}$.

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Remarks:

- Prod'homme's strategy uses estimates on the variance of the number of points.
- Appropriate estimates were difficult to obtain around the edge of the support.

Work with Raphaël Butez (Université de Lille) and David García-Zelada (Sorbonne Université)

Let X be a random locally finite configuration of points in $\mathbb{R}^2$.

For $B \subset \mathbb{R}^2$, $X(B)$ is the number of points of $X \cap B$.

Suppose that X is stationary, i.e. its distribution is translation invariant.

Intensity

$$\exists \lambda > 0 \text{ s.t. } \forall B \in \mathcal{B}(\mathbb{R}^2), \quad \mathbb{E}[X(B)] = \lambda |B|.$$

Definition

X is said to be hyperuniform if

$$\frac{\mathrm{Var}(X(D(0,R)))}{|D(0,R)|} \xrightarrow[R\to\infty]{} 0.$$

Related works:

- Lachièze-Rey and Yogeshwaran (2024): Hyperuniformity and optimal transport of point processes.

- Huesmann and Leblé (2024): The link between hyperuniformity, Coulomb energy, and Wasserstein distance to Lebesgue for two-dimensional point processes.

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In what follows:
$D(0, R)$ will be replaced by B with B a square included in $Q_N = [0, \sqrt{N}]^2$.

Theorem

Fix $p \geqslant 1$. Let $N \geqslant 1$ and $Q_N = [0, \sqrt{N}]^2$. Let X be a random point process on $\mathbb{R}^2$ such that:

1. *There exists a non-increasing function $g : [1, \infty) \to (0, \infty)$ s.t. for any square $B \subset Q_N$ with $|B| \geqslant 1$,*

$$\mathbb{E}[|X(B) - \mathbb{E}[X(B)]|^p] \leqslant |B|^{p/2} g(|B|)^p.$$

2. *There exists $0 < a \leqslant A$ s.t. for every square $B \subset Q_N$,*

$$a|B| \leqslant \mathbb{E}[X(B)] \leqslant A|B|.$$

$$\text{Set} \quad L_N = \sum_{x \in X \cap Q_N} \delta_x \quad \text{and} \quad \bar{L}_N = \frac{L_N(Q_N)}{\mathbb{E}[L_N(Q_N)]} \mathbb{E}[L_N].$$

Then

$$\mathbb{E}[W_p^p(L_N, \bar{L}_N)] \leqslant C_{p,a,A} N \left(1 + g(1) + \int_1^N g(y) \frac{dy}{y}\right)^p.$$

Link with hyperuniformity

Assumption 1 for $p = 2$:

$$\operatorname{Var} X(B) \leqslant |B| g(|B|)^2.$$

- Poisson point process: $g(y) = 1$ and we get

$$\mathbb{E}[W_p^p(L_N, \bar{L}_N)] \leqslant C_{p,a,A} N (\log N)^p.$$

- For determinantal point processes,

$$\mathbb{E}[|X(B) - \mathbb{E}[X(B)]|^p] \leqslant (\operatorname{Var} X(B))^{p/2}.$$

$\Rightarrow$ It suffices to prove Assumption 1 for $p = 2$.

- For hyperuniform point processes, $g(y) \to 0$ as $y \to \infty$.

- Optimal rate of convergence as soon as $\int_1^N g(y)\frac{dy}{y}$ is bounded in N.

Examples

$g(y) = y^{-\varepsilon}$ or $g(y) = \log(y)^{-1-\varepsilon}$ with $\varepsilon > 0$

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Examples

$g(y) = y^{-\varepsilon}$ or $g(y) = \log(y)^{-1-\varepsilon}$ with $\varepsilon > 0$

- In dimension $d \geqslant 3$, Assumption 1 becomes

$$\mathbb{E}[|X(B) - \mathbb{E}[X(B)]|^p] \leqslant |B|^{p\frac{d-1}{d}} g(|B|)^p.$$

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Examples

$g(y) = y^{-\varepsilon}$ or $g(y) = \log(y)^{-1-\varepsilon}$ with $\varepsilon > 0$

- In dimension $d \geqslant 3$, Assumption 1 becomes for $p = 2$

$$\operatorname{Var} X(B) \leqslant |B|^{2\frac{d-1}{d}} g(|B|)^2 = |B| \times |B|^{1-2/d} g(|B|)^2.$$

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Then

$$\mathbb{E}[W_p^p(X_{|Q_N}, \bar{X}_{|Q_N})] \leqslant C_{p,d,a,A} N\Big(1 + g(1) + \int_1^N g(y)\frac{dy}{y}\Big)^p.$$

$\rightsquigarrow$ for Poisson PP: $g(y) = y^{1/d-1/2}$.
$\rightsquigarrow$ for hyperuniform PP: $g(y) = o(y^{1/d-1/2})$.

Thank you for your attention!

A brief argument for the lower bound

- Hölder's inequality: $W_p(\mu_N, \mu) \geqslant W_1(\mu_N, \mu)$ (for probability measures)
- Kantorovich duality: $W_1(\mu_N, \mu) = \sup \left(\int_{\mathbb{R}^d} f d\mu_N - \int_{\mathbb{R}^d} f d\mu \right)$, where the sup is taken over all f 1-Lipschitz functions.
- Set $f : x \in \mathbb{R}^d \mapsto \max\{(c - \|x - x_i\|)_+ ; 1 \leqslant i \leqslant N\}$, where $c > 0$ is to be chosen later:
  - $\rightsquigarrow$ $\int_{\mathbb{R}^d} f d\mu_N = c$
  - $\rightsquigarrow$ $\int_{\mathbb{R}^d} f d\mu \leqslant AN \int_{B_d(0,c)} (c - \|x\|) d\mathrm{Leb}(x) \leqslant \alpha N c^{d+1}$ where A is an upper bound on the p.d.f. of μ.
- choose $c \propto N^{-1/d}$ and $W_1(\mu_N, \mu) \gtrsim N^{-1/d}$.

Inequalities for more general laws

Theorem (Fournier-Guillin 2015)

If all moments of μ are finite,

$$\mathbb{E}[W_p^p(\mu_N,\mu)]^{1/p} \leqslant C_{p,d,\mu} \times \begin{cases} N^{-1/2p} & \text{if } p > \frac{d}{2} \\ \left(\log N\right)^{1/p} N^{-1/2p} & \text{if } p = \frac{d}{2} \\ N^{-1/d} & \text{if } 1 \leqslant p < \frac{d}{2}. \end{cases}$$

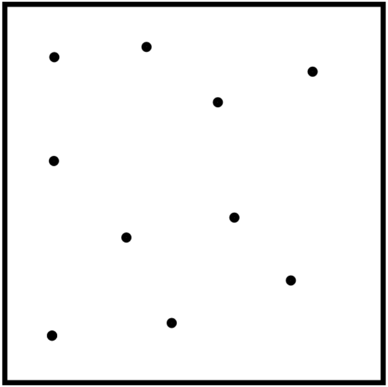
Written for μ with finite moments but more general form in the paper.

These inequalities are (almost) optimal for general distributions.

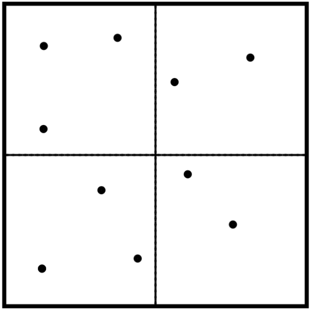
Sketch of the proof

- start from the unit square Q_N: $\nu_0 = \bar{L}_N$.

$$\bar{L}_N = \frac{L_N(Q)}{\mathbb{E}[L_N(Q)]}\mathbb{E}[L_N].$$



- divide Q_N in four equal squares (the set of these squares is denoted by $\mathcal{B}_1$).



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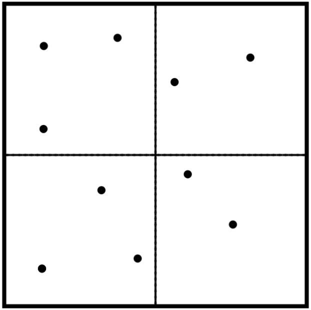
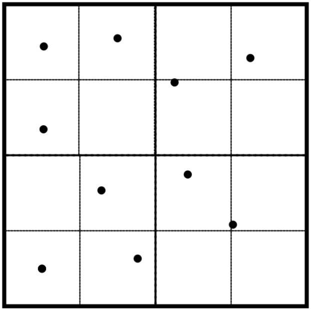


- affect the mass in each subsquare s.t. ν_1 has the same mass as L_N in each subsquare:

$$\nu_1 = \sum_{B \in \mathcal{B}_1} \frac{L_N(B)}{\mathbb{E}[L_N(B)]} \mathbb{E}[L_N]_{|B}.$$

- divide each square in four equal squares (the set of these squares is denoted by $\mathcal{B}_2$).



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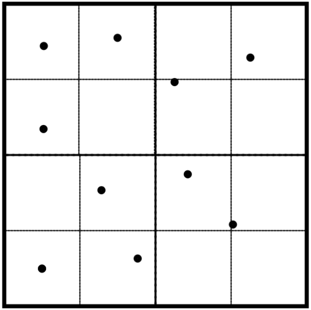

- affect the mass in each subsquare s.t. ν_2 has the same mass as L_N in each subsquare:

$$\nu_2 = \sum_{B \in \mathcal{B}_2} \frac{L_N(B)}{\mathbb{E}[L_N(B)]} \mathbb{E}[L_N]_{|B}.$$

- go on until the subsquares have size $O(1)$: $\nu_0, \dots, \nu_K$, with $K = O(\log_4(N))$.

- At step k, $\mathcal{B}_k$ is a set of 4^k equal squares and

$$\nu_k = \sum_{B \in \mathcal{B}_k} \frac{L_N(B)}{\mathbb{E}[L_N(B)]} \mathbb{E}[L_N]_{|B}.$$

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$$\nu_k = \sum_{B \in \mathcal{B}_k} \frac{L_N(B)}{\mathbb{E}[L_N(B)]} \mathbb{E}[L_N]_{|B}.$$

Note that

$$\mathbb{E}[W_p^p(L_N, \bar{L}_N)]^{1/p} \leqslant \sum_{k=0}^{K-1} \mathbb{E}[W_p^p(\nu_k, \nu_{k+1})]^{1/p} + \mathbb{E}[W_p^p(\nu_K, L_N)]^{1/p}.$$

$\rightsquigarrow$ control each term separately

From ν_k to ν_{k+1}, $k \in \{0, \dots, K-1\}$

Observe that for all $B \in \mathcal{B}_k$, $\nu_k(B) = \nu_{k+1}(B) = L_N(B)$.

Then

$$\mathbb{E}[W_p^p(\nu_k, \nu_{k+1})] \leqslant \sum_{B \in \mathcal{B}_k} \mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1|B})].$$

On a square $B \in \mathcal{B}_k$:

- ν_k has $x \in B \mapsto \frac{L_N(B)}{\mathbb{E}[L_N(B)]}\mathbb{E}[L_N](x)$ for pdf.
- there are 4 squares $B' \in \mathcal{B}_{k+1}$ s.t. $B' \subset B$.
- on each B', ν_{k+1} has $x \in B' \mapsto \frac{L_N(B')}{\mathbb{E}[L_N(B')]}\mathbb{E}[L_N](x)$ for pdf.

"Good event"

Define $G_B = \{L_N(B) \geqslant \frac{1}{2}\mathbb{E}[L_N(B)]\}$.

$\Rightarrow$ The pdf of ν_k is greater than $\frac{a}{2}$, where $a > 0$ is a lower bound on $\mathbb{E}[L_N](x)$.

Lemma (Goldman-Trevisan (2021))

Let R be a square and let μ and λ be two measures on R with equal mass, both absolutely continuous with respect to Lebesgue measure and such that $\inf_R \lambda > 0$. Then for every $p \geqslant 1$,

$$W_p^p(\mu, \lambda) \leqslant \theta_p \frac{(diam R)^p}{(\inf_R \lambda)^{p-1}} \int_R |\mu - \lambda|^p.$$

Then

$$\mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1|B}) \mathbb{1}_{G_B}] \lesssim (\mathrm{diam} B)^p \mathbb{E}\left[\int_B \left| \nu_{k|B} - \nu_{k+1|B} \right|^p \right]$$

$$
\begin{aligned}
&\mathbb{E}\Big[\int_B \Big|\nu_{k|B} - \nu_{k+1|B}\Big|^p\Big] \\
&= \sum_{B' \subset B, B' \in \mathcal{B}_{k+1}} \mathbb{E}\Big[\Big|\frac{L_N(B)}{\mathbb{E}[L_N(B)]} - \frac{L_N(B')}{\mathbb{E}[L_N(B')]}\Big|^p\Big] \int_{B'} \mathbb{E}[L_N](x)^p dx.
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big[\int_B \Big|\nu_{k|B} - \nu_{k+1|B}\Big|^p\Big] \\
&= \sum_{B'\subset B, B'\in\mathcal{B}_{k+1}} \mathbb{E}\Big[\Big|\frac{L_N(B)}{\mathbb{E}[L_N(B)]} - \frac{L_N(B')}{\mathbb{E}[L_N(B')]}\Big|^p\Big] \int_{B'} \mathbb{E}[L_N](x)^p dx.
\end{aligned}
$$

Recall that

$$
\mathbb{E}[|L_N(B) - \mathbb{E}[L_N(B)]|^p] \leqslant |B|^{p/2} g(|B|)^p
$$

and

$$
\mathbb{E}[L_N(B)] \geqslant a|B|.
$$

Suppose g is non-increasing.

Then

$$
\mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1|B})\mathbb{1}_{G_B}] \lesssim (\mathrm{diam} B)^p |B|^{1-p/2} g\Big(\frac{|B|}{4}\Big)^p.
$$

Outside G_B

Use the crude bound

$$W_p^p(\nu_{k|B}, \nu_{k+1,B}) \leqslant (\mathrm{diam} B)^p \nu_k(B).$$

Outside G_B, $\nu_k(B) \leqslant \frac{1}{2}\mathbb{E}[L_N(B)]$.

Then

$$\mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1,B})\mathbb{1}_{G_B^c}] \lesssim (\mathrm{diam} B)^p \mathbb{E}[L_N(B)]\mathbb{P}(G_B^c).$$

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$$\mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1,B})\mathbb{1}_{G_B^c}] \lesssim (\mathrm{diam} B)^p \mathbb{E}[L_N(B)]\mathbb{P}(G_B^c).$$

Note that $\mathbb{P}(G_B^c) = \mathbb{P}(1 - \frac{L_N(B)}{\mathbb{E}[L_N(B)]} > \frac{1}{2})$.

Markov inequality + Assumption 1:

$$\mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1,B})\mathbb{1}_{G_B^c}] \lesssim (\mathrm{diam} B)^p |B|^{1-p/2} g(|B|)^p.$$

Gluing the squares

Recall that

$$\mathbb{E}[W_p^p(\nu_k, \nu_{k+1})] \leqslant \sum_{B \in \mathcal{B}_k} \mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1|B})].$$

Hence

$$\mathbb{E}[W_p^p(\nu_k, \nu_{k+1})] \lesssim \sum_{B \in \mathcal{B}_k} (\operatorname{diam} B)^p |B|^{1-p/2} g\left(\frac{|B|}{4}\right)^p.$$

Recall that

$$\mathbb{E}[W_p^p(\nu_k, \nu_{k+1})] \leqslant \sum_{B \in \mathcal{B}_k} \mathbb{E}[W_p^p(\nu_{k|B}, \nu_{k+1|B})].$$

Hence

$$\mathbb{E}[W_p^p(\nu_k, \nu_{k+1})] \lesssim \sum_{B \in \mathcal{B}_k} (\mathrm{diam} B)^p |B|^{1-p/2} g\left(\frac{|B|}{4}\right)^p.$$

For $B \in \mathcal{B}_k$, $|B| = \frac{N}{4^k}$ and $\mathrm{diam}(B) = \frac{\sqrt{2N}}{2^k}$. Moreover $\#\mathcal{B}_k = 4^k$.
Then

$$\mathbb{E}[W_p^p(\nu_k, \nu_{k+1})] \lesssim Ng\left(\frac{N}{4^{k+1}}\right)^p.$$

$$\mathbb{E}[W_p^p(\nu_0, \nu_K)]^{1/p} \lesssim N^{1/p} \sum_{k=0}^{K-1} g\left(\frac{N}{4^{k+1}}\right) \lesssim N^{1/p} \int_1^N g(y) \frac{dy}{y}.$$

$$\mathbb{E}[W_p^p(\nu_0,\nu_K)]^{1/p} \lesssim N^{1/p} \sum_{k=0}^{K-1} g\Big(\frac{N}{4^{k+1}}\Big) \lesssim N^{1/p} \int_1^N g(y)\frac{dy}{y}.$$

Last step:
Crude bound on $W_p(\nu_{K|B}, L_{N|B})$ for every $B \in \mathcal{B}_K$ and gluing the squares:

$$\mathbb{E}[W_p^p(\nu_K, L_N)] \lesssim N.$$

Hence

$$\mathbb{E}[W_p^p(L_N,\bar{L}_N)] \lesssim N\Big(1 + g(1) + g(N) + \int_1^N g(y)\frac{dy}{y}\Big)^p.$$