TWO POINT FUNCTION FOR CRITICAL POINTS

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F smooth Gaussian field.

Number and positions of critical points of F are important qualitative descriptor. The geometry of nodal lines, nodal domains, level curves and excursion sets are closely related to the set of critical points.

Set of critical points as a point process on $\mathbb{R}^2$.

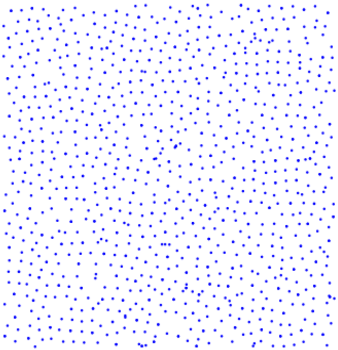


Figure: Critical points of a Berry's Random Plane Wave Model [Beliaev-C.-Wigman].

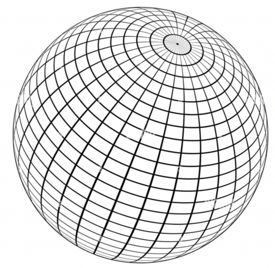
Random spherical harmonics

$\mathbb{S}^2$ two-dimensional unit sphere with the round metric

$\Delta_{\mathbb{S}^2}$ spherical Laplacian

$\Delta_{\mathbb{S}^2} f + \lambda_\ell f = 0$ Helmholtz equation

$-\lambda_\ell = -\ell(\ell+1)$, $\ell \in \mathbb{N}$ eigenvalues

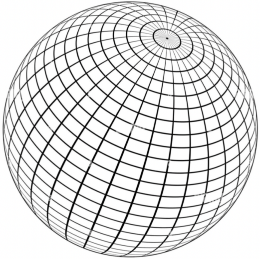


the eigenspace corresponding to $\lambda_\ell = \ell(\ell+1)$ is the space of degree-ℓ spherical harmonics of dimension $2\ell+1$, for any eigenvalue choose an arbitrary L^2-orthonormal basis $\{Y_{\ell m}(\cdot)\}_{m=-\ell,\dots,\ell}$ and consider *random Gaussian eigenfunctions*

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \qquad x \in \mathbb{S}^2,$$

$\{a_{\ell m}\}$ are zero-mean, independent Gaussian.

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$$



Standardised s.t. $\mathrm{Var}(f_\ell(x)) = 1$.

From the addition formula for Legendre polynomials the covariance kernel is

$$K_\ell(x,y) = \mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos d(x,y)),$$

P_ℓ Legendre polynomial, $d(x,y) = \arccos\langle x, y\rangle$ geodesic distance on the sphere.

$\implies$ isotropic field, alternatively, this could be seen from the invariance of the eigenspace and L^2 norm under rotations.

Number of **critical points**: $\mathcal{N}_\ell^c = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0\}$

Number of **critical values** in $I \subseteq \mathbb{R}$: $\mathcal{N}_\ell^c(I) = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0, f_\ell(x) \in I\}$.

[C.-Marinucci-Wigman] show that as $\ell \to \infty$

The expected number of critical values behaves like

$$\mathbb{E}[\mathcal{N}_\ell^c(I)] = \frac{2}{\sqrt{3}}\ell^2 \int_I \frac{\sqrt{3}}{\sqrt{8\pi}}(2e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}}\,dt + O(1),$$

the constant in the $O(\cdot)$ term is universal, i.e. the integral of the error term on any interval I is uniformly bounded by its value when $I = \mathbb{R}$.

The investigation of the **asymptotic variance** is more challenging

$$\text{Var}(\mathcal{N}_\ell^c(I)) = \ell^3[\nu^c(I)]^2 + O(\ell^{5/2}),$$

$$\nu^c(I) = \int_I \frac{1}{\sqrt{8\pi}}[2 - 6t^2 - e^{t^2}(1 - 4t^2 + t^4)]e^{-\frac{3}{2}t^2}\,dt, \qquad \nu^c(\mathbb{R}) = 0.$$

[C.-Wigman] show that for $I = \mathbb{R}$

$$\text{Var}(\mathcal{N}_\ell^c) = \frac{1}{3^3\pi^2}\ell^2 \log \ell + O(\ell^2).$$

Similar results hold for extrema and saddles.

Proof: (approximate) Kac-Rice formula for moments.

Random Plane Wave

Random Plane Wave is a stationary Gaussian field F in $\mathbb{R}^2$ with covariance kernel

$$\begin{aligned} K(x,y) &= \mathbb{E}[F(x)F(y)] \\ &= J_0(|x-y|) \\ &= \sqrt{\frac{2}{\pi|x-y|}}\cos\left(|x-y|-\frac{\pi}{4}\right)+O(|x-y|^{-3/2}), \quad x,y\in\mathbb{R}^2. \end{aligned}$$

- point-wise covariance goes to zero as points move away from each other
- but the rate is quite slow
- the covariance kernel is as oscillating function
- RPW is a stationary field i.e. $F(\cdot)$ and $F(\cdot+z)$ have the same distribution for every z or $K(x,y)=K(x-y)$

The corresponding spectral measure ρ is the normalized arc-length on the unit circle. Since the support of ρ is on the unit circle, the field is an eigenfunction of the Laplacian

$$K(x)=\int_{\mathbb{R}^2} e^{-2\pi i x\cdot t}\rho(dt).$$

One can think RPW as (the limiting ensemble of) a random linear combination of plane waves with the same energy $E = k^2$ travelling in all possible directions

$$u_k(x) = \lim_{J \to \infty} \frac{1}{\sqrt{J}} \Re \left(\sum_{j=1}^{J} e^{k \langle \theta_j , x \rangle + \phi_j} \right)$$

θ_j random directions drawn uniformly on the unit circle, $\phi_j \in [0, 2\pi)$ random phases.

In this sense, one can think RPW is a natural notion of a 'typical' eigenfunction of the Laplacian in $\mathbb{R}^2$.

[Berry 1977] proposed to compare eigenfunctions with eigenvalue λ to a 'typical' instance of an isotropic, monochromatic random wave with wavenumber $k = \sqrt{\lambda}$ (now called RPW).

Berry conjectured that *high energy behaviour of deterministic eigenfunctions* on generic chaotic surfaces is universal and has statistically the same behaviour as Random Plane Waves. Nodal lines (nodal length, boundary intersections, intersections with a test curve...) of RPW should model nodal lines of honest eigenfunctions.

This vague relation is subject to many numerical tests with very positive outcomes.

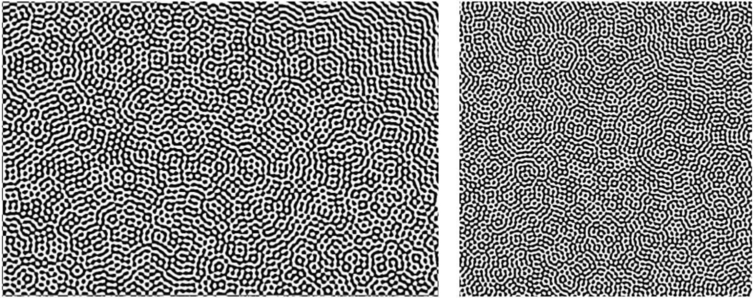


Figure: Nodal domains. Left: eigenfunction of a quarter of the stadium. Right: RPW [Bogomolny and Schmit 2007]

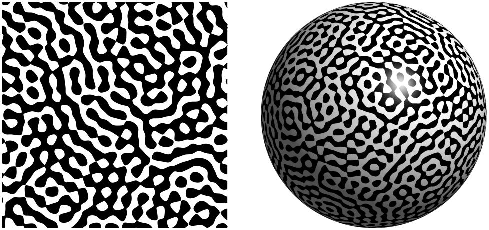


Figure: RPW (left) and random spherical harmonic of degree 100 (right) [Beliaev 2022]. RPW is the scaling limit of random spherical harmonics.

$x_0 \in \mathbb{S}^2$, exponential map from the tangent plane to the sphere: $\exp_{x_0} : T_{x_0}\mathbb{S}^2 \to \mathbb{S}^2$
rescale by $\ell \sim \sqrt{\lambda}$ so the wavelength becomes of order 1

$$g_\ell(x) := f_\ell(\exp_{x_0}(x/\ell)),$$

$$\mathbb{E}[g_\ell(x)g_\ell(y)] = P_\ell(\cos\theta(\exp_{x_0}(x/\ell), \exp_{x_0}(y/\ell))).$$

Since the exponential map is almost isometry near x_0, the spherical distance between images is almost the distance between the points, hence uniformly in x and y the covariance behaves as

$$P_\ell\left(\cos\frac{|x-y|}{\ell}\right) \to J_0(|x-y|)$$

where the limit follows from Hilb's asymptotics for Legendre polynomials.

Expected number of critical points of RPW

Number of critical points in a ball of radius ρ

$$\mathcal{N}^c_\rho = \#\{x \in B(\rho) : \nabla F(x) = 0\}.$$

Kac-Rice method to count the zeros of the map $x \to \nabla F(x)$. If $\nabla F(x)$ is nonsingular for all $x \in B(\rho)$

$$\mathbb{E}[\mathcal{N}^c_\rho] = \int_{B(\rho)} \phi_{\nabla F(x)}(0) \cdot \mathbb{E}[|\det H_F(x)| \,\big|\nabla F(x) = 0]dx.$$

F *stationary* $\implies$ first and second order derivatives are independent at every fixed point.

F *isotropic* $\implies$ $\mathbb{E}[\mathcal{N}^c_\rho] = \mathrm{Vol}(B(\rho))\frac{1}{2\pi\sqrt{3}}$.

Compare critical points of RPW with two well known translation invariant processes.

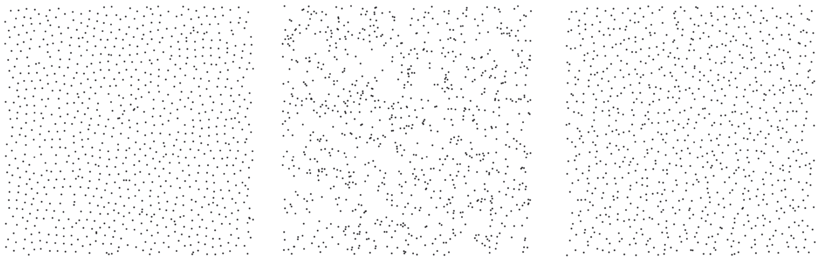


Figure: Left: Critical points of a Random Plane Wave. Centre: Poisson point process. Right: Ginibre ensemble.

The number of critical points in a square of side-length n is $c \cdot n^2$ where $c = \frac{1}{2\sqrt{3}\pi}$ is the natural intensity of critical points.

The other two point processes are rescaled to have the same intensity.

Fluctuations of the number of critical points

Fluctuations of the number of points in a square depend a lot on the point process.

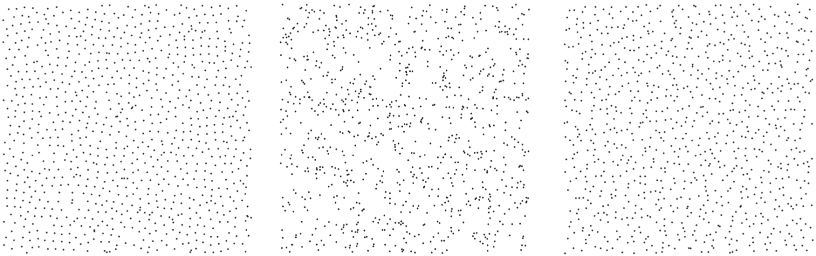


Figure: Left: Critical points of a Random Plane Wave. Centre: Poisson point process. Right: Ginibre ensemble.

- Poisson: variance is asymptotic to $c \cdot n^2$.
- Ginibre ensemble: variance is of order n.
- When the covariance function decays sufficiently rapidly, the *long range* asymptotics of $K_2(r)$, $r \to \infty$ yields the asymptotic variance of the number of critical points in large balls. From random spherical harmonics (equivalent to RPW in the scaling limit), variance for the number of critical points scales like $n^2 \log n$.

attraction/repulsion

The *short range* asymptotics of $K_2(r)$, $r \to 0$ yields the asymptotic second factorial moment of the number of critical points in a small ball.

Informally, for ρ small,

$$\mathbb{P}(\text{one critical points in } B(\rho)) \approx \rho^2$$
$$\mathbb{P}(\text{two critical points in } B(\rho)) \approx \iint_{B(\rho)\times B(\rho)} K_2(x,y)dxdy.$$

- If $K_2(r) \to \infty$ as $r \to 0$, then

  $$\{\mathbb{P}(\text{one critical points in } B(\rho))\}^2 \ll \mathbb{P}(\text{two critical points in } B(\rho))$$

  i.e critical points attract each other.

- If $K_2(r) \to 0$ as $r \to 0$, then

  $$\mathbb{P}(\text{two critical points in } B(\rho)) \ll \{\mathbb{P}(\text{one critical points in } B(\rho))\}^2$$

  i.e. critical points repel each other.

Theorem (Beliaev C. Wigman)

As $\rho \to 0$

$$\mathbb{E}[\mathcal{N}^c_\rho(\mathcal{N}^c_\rho - 1)] = \frac{1}{2^5 3\sqrt{3}}\rho^4 + O(\rho^6).$$

Proof

Second factorial moment of the number of critical points

$$\mathbb{E}[\mathcal{N}^c_\rho(\mathcal{N}^c_\rho - 1)] = \int\limits_{B(\rho)\times B(\rho)} K_2(x,y)dxdy,$$

where

$$K_2(x,y) = \phi_{(\nabla F(x), \nabla F(y))}(0,0) \cdot \mathbb{E}\left[|\det H_F(x) \cdot \det H_F(y)| \,\middle|\, \nabla F(x) = \nabla F(y) = 0\right]$$

we evaluate the entries of the covariance matrix $\Delta(x,y)$ of $(\nabla^2 F(x), \nabla^2 F(y))$ conditioned on $\nabla F(x) = \nabla F(y) = 0$.

F *isotropic* $\implies$ $K_2(x,y)$ is a function of the Euclidean distance $r = \|x-y\|$, we reduce the number of variables and make explicit computations. There are no cancellations suggesting the same asymptotic behaviour in the generic case, see [Ladgham-Lachiéze-Rey 2023].

$$K_2(r) = \frac{1}{(2\pi)^5\sqrt{\mathrm{det}(A(r))}} \int_{\mathbb{R}^6} |x_1x_3 - x_2^2| \cdot |x_4x_6 - x_5^2|$$
$$\times \frac{1}{\sqrt{\mathrm{det}(\Delta(r))}} \exp\left\{-\frac{1}{2}\mathbf{x}^t\Delta(r)^{-1}\mathbf{x}\right\} d\mathbf{x}.$$

Our aim is to study the asymptotic behaviour of K_2 in the vicinity of $r = 0$. It is useful to transform the coordinates to rewrite the integrand in terms of the standard Gaussian vector.

For every $r > 0$, $\Delta(r)$ is symmetric, we diagonalise $\Delta(r)$: $\Delta(r) = P^t(r)\Lambda(r)P(r)$.

Taylor expand around the origin the entries in $\Lambda(r)$, $Q(r) = P^t(r)$ and det$(A(r))$.

$$\begin{aligned}
K_2(r) &= \frac{1}{(2\pi)^5\sqrt{\mathrm{det}A(r)}} \int_{\mathbb{R}^6} \big|f(r,\mathbf{z})\big| \exp\left\{-\frac{1}{2}\sum_{i=1}^{6} z_i^2\right\} d\mathbf{z} \\
&= \frac{1}{\pi^5 2\sqrt{3}r^2 + O(r^4)}\Big[\frac{r^2}{384}\int_{\mathbb{R}^6} z_4^2 z_6^2 \exp\left\{-\frac{1}{2}\sum_{i=1}^{6} z_i^2\right\} d\mathbf{z} + O(r^4)\Big] \\
&= \frac{1}{96\sqrt{3}\,\pi^2} + O(r^2).
\end{aligned}$$

The asymptotic behaviour of the variance, dominated by the expectation (and hence less useful) can be easily obtained.

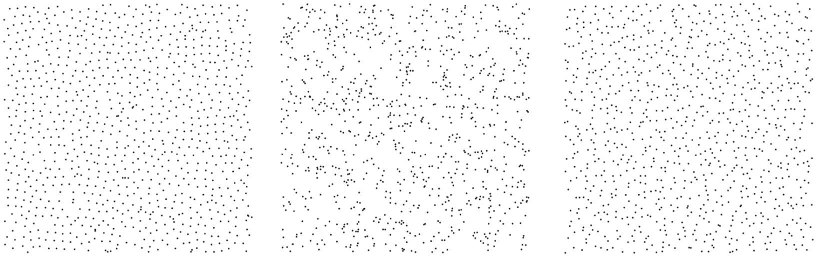


Figure: Left: Critical points of a Random Plane Wave. Centre: Poisson point process. Right: Ginibre ensemble.

- Ginibre ensemble: $\mathbb{P}(\text{two points in } B(\rho)) \approx \rho^6$, points repel each other.
- For Poisson process (or a finite collection of independent points) on the plane

$$\mathbb{P}(\text{two points in } B(\rho)) = \frac{1}{2}(c\pi\rho^2)^2 e^{-c\pi\rho^2} \approx \rho^4$$

- Critical points of RPW: $\mathbb{P}(\text{two points in } B(\rho)) \approx \rho^4$ no repulsion nor attraction.

The minor difference (by a constant) does not explain the big difference in the appearance of Poisson and critical points (highly regular).

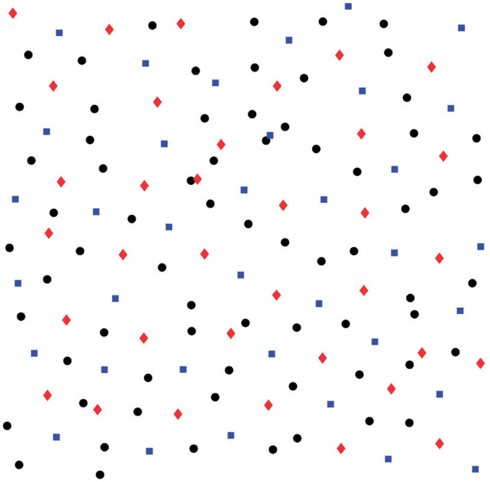


Figure: Different types of critical points: diamonds are local maxima, squares are local minima, and discs are saddles.

Theorem (Beliaev C. Wigman)

As $\rho \to 0$

$$\begin{aligned}
&\mathbb{E}[\mathcal{N}_\rho^{max}(\mathcal{N}_\rho^{max}-1)] = \mathbb{E}[\mathcal{N}_\rho^{min}(\mathcal{N}_\rho^{min}-1)] = O(\rho^7\log(1/\rho)),\\
&\mathbb{E}[\mathcal{N}_\rho^{e}(\mathcal{N}_\rho^{e}-1)] = O(\rho^7\log(1/\rho)),\\
&\mathbb{E}[\mathcal{N}_\rho^{saddle}(\mathcal{N}_\rho^{saddle}-1)] = O(\rho^7\log(1/\rho)),\\
&\mathbb{E}[\mathcal{N}_\rho^{max}\mathcal{N}_\rho^{min}] = O(\rho^{12}),\\
&\mathbb{E}[\mathcal{N}_\rho^{e}\mathcal{N}_\rho^{saddle}] = \frac{1}{2^6\,3\,\sqrt{3}}\,\rho^4 + O(\rho^6).
\end{aligned}$$

Proof

Evaluate the two-point function modified for the respective types of critical points: integration over a proper subset of $\mathbb{R}^6$.

There is no evidence that these estimates are sharp.

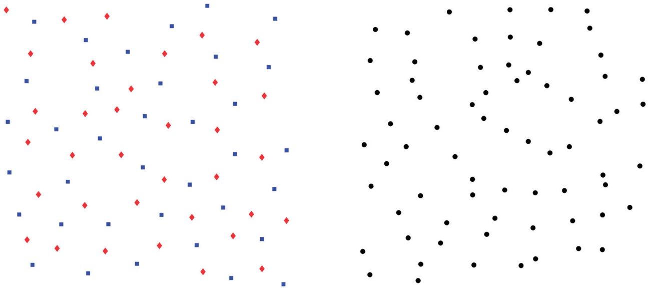


Figure: Left: Extrema only. Right: Saddles only.

Both processes exhibit strong repulsion:

$$\mathbb{P}(\text{at least } 2 \text{ points in } \mathcal{B}(\rho)) \leq \rho^7 \log 1/\rho.$$

smaller than the decay rate for the Ginibre ensemble.

The apparent *rigid* structure that is observed for the critical points comes from the regularity of both these point processes.

All clustering comes from the probability that, after overlapping, a point in one process is close to a point in another process.

Isotropic stationary Gaussian fields

Beliaev-C.-Wigman: critical points of a smooth, stationary, isotropic Gaussian field neither repel nor attract each other.

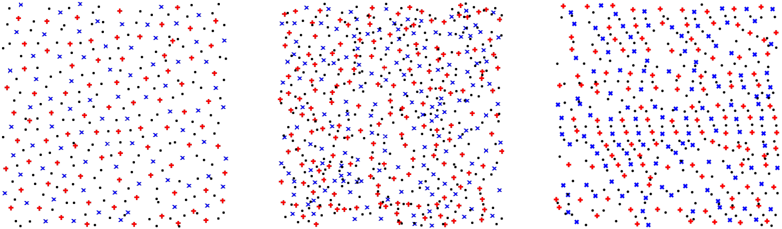


Figure: Critical points for the random plane wave (left), Bargmann-Fock field (center) and an anisotropic field. Minima and maxima are red and blue, critical points are black.

Theorem (Beliaev-C.-Wigman)

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a nonconstant stationary isotropic Gaussian random field, and assume that F is a.s. $C^{4+\epsilon}(\mathbb{R}^2)$ for some $\epsilon > 0$. Then the 2-point correlation function admits the following expansion around the origin

$$K_2(r) = \frac{\sqrt{3}}{\pi^2} \frac{A(g_4, g_6)^2 + B(g_4, g_6)^2}{\sqrt{\phi(g_4, g_6)}} + O_{r\to 0}(r^2).$$

Where g_{2k} (spectral moments) are given by

$$\mathbb{E}[F(x) \cdot F(y)] = C_F(\|x - y\|) = C_F(r) = 1 - g_2 r^2 + g_4 r^4 - g_6 r^6 + O(r^8),$$

by rescaling F we may assume without loss of generality that $g_2 = 1$.

We have $A(g_4, g_6)^2 + B(g_4, g_6)^2 = 0$ for real strictly positive $g_4, g_6 \iff g_4^2 = \frac{5}{2} g_6$.
By Cauchy-Schwarz

$$g_4^2 = \frac{(2\pi)^8}{(4!)^2} \left(\int x_1^4 d\rho\right)^2 \leq \frac{(2\pi)^8}{(4!)^2} \int x_1^6 d\rho \int x_1^2 d\rho = \frac{5}{2} g_2 g_6 = \frac{5}{2} g_6.$$

The equality holds if and only if ρ is the δ-measure at the origin i.e. F is a (random) constant. This means that the leading term is non-zero for non-degenerate F.

Azaïs-Delmas: smooth stationary and isotropic Gaussian field $\{X(t) : t \in \mathbb{R}^d\}$, $d = 1$ repulsion, $d > 2$

- correlation function between critical points: $K_2(r) \approx r^{2-d} \implies$ attraction
- correlation function between maxima and minima: for all $\epsilon > 0$ there exists a constant $K(\epsilon) > 0$ such that $K_2(r) \leq K(\epsilon) r^{2d-1-\epsilon} \implies$ repulsion
- correlation function between maxima (respectively minima): for all $\epsilon > 0$ there exists a constant $K(\epsilon) > 0$ such that $K_2(r) \leq K(\epsilon) r^{5-d-\epsilon} \implies$ strong repulsion
- correlation function between critical points with adjacent indexes: $K_2(r) \approx r^{2-d}$, attraction between critical points due to critical points with adjacent indexes.

Proof: asymptotical evaluation of the two-point function around the diagonal with technique from [Fyodorov 2004]

$$\nabla^2 X(t) \overset{law}{=} \sqrt{\frac{2\lambda_4}{3}} (G_d - \Lambda I_d)$$

where G_d is a size d GOE matrix and $\Lambda \sim \mathcal{N}(0, 1/2)$ independent of G_d. Explicit expression for the joint density of GOE eigenvalues is exploited.

Higer moments

Beliaev-McAuley-Muirhead: finiteness of the third moment for the number of critical points of a regular non-degenerate Gaussian field taking values in $\mathbb{R}$ (by proving the integrability of the Kac density). The proof relies on a technical Taylor expansion near the singularity of the Kac density.

Gass-Stecconi and Ancona-Letendre: prove finiteness of the p-th moment of the number of critical points on a compact subset for a large class of smooth Gaussian field (assuming regularity and non-degeneracy of the Taylor polynomial expansion up to order p).

Proof: relies on Kergin interpolation, which allows to pass from a general Gaussian field to a random polynomial field, for which the finiteness of the number of critical points is a consequence of Bézout's theorem.

This result is not applicable to Berry random wave model (partial derivatives at a point $x \in \mathbb{R}^d$ form a degenerate Gaussian vector).

C.-Stecconi: asymptotic behaviour of higher moments, this is work in progress!

Thank you!

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