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Markets, Portfolios and Arbitrage

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Outline

Motivation

- Let $\{S_t\}_{t\in\mathbb{N}_0}$ be the daily price process of a risky asset A.
- We view $\{S_t\}_{t\in\mathbb{N}_0}$ as a (discrete) stochastic process on some filtred probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{N}_0})$ encorporating the states of the world.
- At a given day t, a financial analyst wants to determine the fair price S_{t+1} of A.
- Since it is risky, the expected return R_t of A should exceed the risk-free rate r_t, i.e. $\mathbb{E}[1 + R_t|S_t] = (1 + r_t)(1 + \mu)$ for some $\mu > 0$
- This implies $\mathbb{E}[R_t|S_t] \approx r_t + \mu$. We can consider μ as a 1-day risk premium.

Motivation

- To price A correctly, the analyst has to consider $\mathbb{E}\left[\frac{1}{1+R_t}S_{t+1}|S_t\right] = S_t$.
- $\rightsquigarrow$ Problem: The left hand side is complicated to evaluate.
- $\rightsquigarrow$ Idea: Perform a transformation in order to eliminate the risk premium μ.
- Ideally find some probability $\tilde{P} \sim P$ describing the odds of some "risk-neutral world "where $\mathbb{E}\left[\frac{1}{1+r_t}S_{t+1}|S_t\right] = S_t$.

Defintion of Continous time market

- let $B(t) = (B_1(t), \dots, B_m(t))$ be m-dimensional Brownian motion, $0 \leq t \leq T$ on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $\mathbb{F}^m = \{\mathcal{F}_t^m \; \forall t \in [0, T] : \mathcal{F}_t^m = \sigma(B_s : 0 \leq s \leq t)\}$ represents the flow of information generated by $\{B(t)\}_{t \in [0,T]}$.
- A market is an $\mathcal{F}_t^m$-adapted $(n+1)$-dimensional Ito-process $X(t) = (X_0(t), X_1(t), \dots X_n(t))$ such that

$$dX_0(t) = \rho(t, \omega) X_0(t) dt; X_0(0) = 1$$

and

$$dX_i(t) = \mu_i(t, \omega) + \sigma_i(t, \omega) dB_t^m; X_i(0) = x_i,$$
$$\text{for } i = 1, \dots, n$$

where $\mu(t, \omega) \in \mathbb{R}^{n \times 1}, \sigma(t, \omega) \in \mathbb{R}^{n \times m}$ meet the existence conditions and $\rho(t, \omega)$ is bounded.

Portfolio in the market

- A portfolio $\theta(t) = (\theta_0(t,\omega), \theta_1(t,\omega) \ldots, \theta_n(t,\omega))$ in the market $\{X(t)\}_{t\in[0,T]}$ is a (t,ω)-measurable and $\mathcal{F}_t^m$-adapted stochastic process for $0 \leq t \leq T$.
- The value at time t of a portfolio $\theta(t)$ is $V(t,\omega) = \theta(t,\omega) \cdot X(t,\omega)$, $\forall t \in [0, T]$.
- The gain process of the portfolio $\theta(t)$ is defined by

$$G(t) = \int_0^t \mu(s) \cdot \theta(s)\mathrm{d}s + \int_0^t \sigma(s)\theta(s)\mathrm{d}B.$$

In order for G to be well defined, we require that $\int_0^t |\mu(s)\theta(s)|\mathrm{d}s < \infty$ and $\int_0^t ||\sigma(s)\theta(s)||^2\mathrm{d}s < \infty$ a.s.

Selffinancing strategy and normalized Market

- A Portfolio $\theta(t)$ is called self-financing if $V(t) = V(0) + G(t)$ or $dV(t) = dG(t)$.
- Equivalently we can write $dV(t) = \theta(t)dX(t)$.
- Let $X_0(t) = exp(\int_0^t \rho(s,\omega)dt) > 0$, then the Process $\overline{X}(t) = \left(1, \frac{X_1(t)}{X_0(t)}, \dots, \frac{X_n(t)}{X_0(t)}\right)$ is called the normalized market.
- We think of the risk-free asset as a bank account paying interest with return rate $\rho(t,\omega)$.

Numeraire Invariance

- The portfolio $\theta(t)$ is self-financing with respect to $\{X(t)\}_{t\in[0,T]}$ if and only if it is self-financing with respect to $\{\overline{X}(t)\}_{t\in[0,T]}$.
- Proof:
  - $\overline{V}_t(t) = \theta(t)\overline{X}(t) = \theta(t)\xi(t)X(t) = \xi(t)V(t)$
  - It follows with Ito-lemma

$$
\begin{aligned}
d\overline{V}(t) &= \xi(t)dV(t) + V(t)d\xi(t) \\
&= \xi(t)\theta(t)dX(t) - \theta(t)X(t)\rho(t)\xi(t)dt \\
&= \xi(t)\theta(t)(dX(t) - \rho(t)X(t)dt) \\
&= \theta(t)d\overline{X}(t).
\end{aligned}
$$

Doubling strategy and Admissible Portfolio

- is imposing self-financing condition on a portfolio sufficient for a consistent market model?
- Example :
  - let $\{B(t)\}_{t\in[0,T]}$ be a one dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
  - $X_0 \equiv 1 \ \forall t \in [0, T]$.
  - $dX_1(t) = X_1(t)dB(t)$, $X_1(0) = 1$.
  - let $\tau := inf\{t : \int_0^t (T-s)^{-1/2} dB(s) = \alpha\}$ for some $\alpha > 0$.
  - τ is a stopping time such that $0 < \tau < T$ a.s.
  - Set $\theta_1(t) = \frac{1}{X_1(t)\sqrt{T-t}}$ if $0 \leq t \leq \tau$ and 0 else.
  - Set $\theta_0(t) = -\theta_1(t)X_1(t) + \int_0^t \theta_1(s)dX_1(s)$ for $t \in [0, T]$.
  - $\theta(0).X(0) = 0$ and $\theta(T).X(T) = \alpha$ a.s.

Doubling strategy and Admissible Portfolio

- Despite the natural assumption on the prices dynamics, we can reach any value α without any initial investment.
- Further restrictions on the portfolio are needed.
- A portfolio $\theta(t)$ is called admissible if it satisfies the condition for the existence of the gain process and its value process $V^{\theta}(t)$ is lower bounded , i.e. $V^{\theta}(t,\omega) \geq -K$ for some real number $K > 0$ and for a.a. $(t,\omega) \in [0, T] \times \Omega$.

Arbitrage

- An admissible portfolio $\theta(t)$ is called an arbitrage in the market $\{X(t)\}_{t\in[0,T]}$ if $V^\theta(t)$ satisfies $V^\theta(0) = 0$ and $V^\theta(T) \geq 0$ a.s. and $P(V^\theta(T) > 0) > 0$.
- An admissible portfolio $\theta(t)$ is an arbitrage for$\{X(t)\}_{t\in[0,T]}$ if and only if it is an arbitrage for $\{\overline{X(t)}\}_{t\in[0,T]}$.
- Arbitrage is a sign of lack of equelibrium in the market.
- In financial markets arbitrage dont survive long time: supply and demand eliminate it.
- Arbitrage can be used to determine the fair price of financial assets.

Equivalent martingale Measure and arbitrage freeness

If there is a probability measure Q on $(\Omega, \mathbb{F})$ such that $P \sim Q$ and that normalized price process $\{\overline{X}_t\}_{t\in[0,T]}$ is a (local)martingale with respect to Q then $\{X_t\}_{t\in[0,T]}$ admits no arbitrage .

Proof:

- Let $\theta(t)$ be an arbitrage for $\{\overline{X}_t\}_{t\in[0,T]}$.
- $\overline{V}^{\theta}(t)$ is a lower (local)martingale and thus a supermartingale with respect to Q.
- $\mathbb{E}_Q[V^{\theta}(T)] \leq V^{\theta}(0) = 0$.
- $V^{\theta}(T, \omega) \geq 0$ Qa.s. and $Q(V^{\theta}(T) > 0) > 0$ since $P \sim Q$ which Implies that $\mathbb{E}_Q[V^{\theta}(T)] > 0$.

Equivalent martingale measure and arbitrage freeness

- In discret time setting the arbitrage freeness insures the existence of an equivalent martingale measure.
- in the continuous time setting we have to settle with the following

Theorem

Let $\hat{X}(t) = \{X_1(t), \ldots, X_n(t)\}$. The market $\{X_t\}_{t \in [0,T]}$ admits no arbitrage if and only if there exists a process $u(t,\omega) \in \mathcal{V}^m(0,T)$ satisfying

- $\mathbb{E}[exp(\frac{1}{2}\int_0^T \|u(t,\omega)\|^2 dt)] < \infty$ *(Novikov's condition)*
- $\sigma(t,\omega)u(t,\omega) = \rho(t,\omega)\hat{X}(t,\omega) - \mu(t,\omega)$ *for a.a.* (t,ω) *in* $[0,T] \times \Omega$.

- Before proving the theorem we need to discuss the Girsanov theorem.

The Girsanov theorem

Theorem

Let $u(t,\omega) \in \mathcal{V}^m(0,T)$ satisfy the Novikov's condition and $\xi(t) := exp(\int_0^t u(s,\omega)dB(t) - \frac{1}{2}\int_0^t \|u(s,\omega)\|^2 ds)$. The process $\tilde{B}(t) := B(t) - \int_0^t u(s,\omega)ds$ is then an $\mathcal{F}_t^m$-adapted m-dimensional Brownian motion under a new probability measure $\tilde{P} \sim P$ on $(\Omega, \mathcal{F})$ such that $\tilde{P}(A) = \mathbb{E}^P[\mathbf{1}_A.\xi(T)]$ for all $A \in \mathcal{F}$.

Discussion of the Girsanov theorem

- The theorem states that $d\tilde{B}(t)$ is obtained by subtracting a drift term from $dB(t)$.
- Both $\{B(t)\}_{t\in[0,T]}$ and $\{\tilde{B}(t)\}_{t\in[0,T]}$ are Brownian motions and thus do not have any drift.
- if $dB(t)$ models the random infinitesimal increments of a given dynamical system then $d\tilde{B}(t)$ can represent the unpredictable infinitesimal errors if we switch from P to $\tilde{P}$.

Arbitrage freeness : The proof

"$\Rightarrow$"

- Assume that $X(t)$ is normalized,i.e. that $\rho = 0$
- Define $\tilde{P}$ on $\mathcal{F}$ as in the Girsanov Theorem.
- $\tilde{B}(t) = B(t) - \int_0^t u(s, \omega)ds$ is a $\tilde{P}$-Brownian motion and $\tilde{P} \sim P$.
- $dX(t) = \sigma d\tilde{B}(t)$ and $X(t)$ is a (local)$\tilde{P}$-Martingale.
- There is an equivalent martingale measure which means that the market is arbitrage free.

Arbitrage freeness : the proof

"$\Leftarrow$"

- Let $A_t = \{\omega : \sigma(t,\omega)u(t,\omega) = -\mu(t,\omega)$ has no solutions$\}$.
- $A_t = \{\omega : \exists a(t,\omega)$ with $\sigma^T(t,\omega)a(t,\omega) = 0$ and $a(t,\omega) \cdot \mu(t,\omega) \neq 0\}$.
- Define $\theta_i(t,\omega) = \text{sign}(a(t,\omega) \cdot \mu(t,\omega))a_i(t,\omega)$ for $\omega \in A_t$ and 0 else for $i = 1, ..., n$.
- choose $\theta_0(t,\omega)$ in such a way that makes $\theta(t)$ self-financing.
- $V^\theta(t,\omega) \geq V^\theta(0)$ for $\forall t \in [0,T]$. Hence $\mathbf{1}_{A_t} = 0$ for a.a. $(t,\omega) \in \Omega \times [0,T]$.

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