Archimedean Copulas

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1 Definitions

Before we give the formal construction of Archimedean copulas, we want to motivate the definition of this class of copulas. By elementary probability theory we know that two continuous random variables X and Y, with joint distribution function H and margins F and G are independent if and only if $H(x,y) = F(x)G(y)$ for all $x, y \in \overline{\mathbb{R}}$. Now there are families of copulas that satisfy a property that looks similar. Remember the Ali-Mikhail-Haq family of copulas given by

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}$$

for $\theta \in [-1,1]$. This family was constructed in a way that

$$\frac{1-C_\theta(u,v)}{C_\theta(u,v)} = \frac{1-u}{u} + \frac{1-v}{v} + (1-\theta)\frac{1-u}{u} \cdot \frac{1-v}{v}$$

holds. So defining $\lambda(t) = 1 + (1-\theta)(1-t)/t$, this can be written as

$$\lambda(C_\theta(u,v)) = \lambda(u)\lambda(v).$$

Going one step further, if we define $\varphi(t) = -\log(\lambda(t))$, we get

$$\varphi(C_\theta(u,v)) = \varphi(u) + \varphi(v).$$

Maybe there are more copulas satisfying

$$\varphi(C(u,v)) = \varphi(u) + \varphi(v),$$

for a function φ. If we define an appropriate inverse $\varphi^{[-1]}$ we can solve this expression for C, so we have a copula fulfilling

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)). \tag{1.1}$$

This leads us to the following definition. Why we expect φ to have these specific properties as stated below, becomes clear later.

Definition 1.1 Let $\varphi : \mathbf{I} \to [0,\infty]$ be continuous, strictly decreasing and such that $\varphi(1) = 0$. The pseudo-inverse $\varphi^{[-1]} : [0,\infty] \to \mathbf{I}$ is defined as

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \le t \le \varphi(0) \\ 0, & \varphi(0) \le t \le \infty \end{cases}.$$

Let us first note some useful properties of the pseudo-inverse.

Lemma 1.2 (Properties of pseudo-inverse) Let $\varphi^{[-1]}$ be defined as above. Then

(i) $\varphi^{[-1]}$ is continuous, non-increasing on $[0,\infty]$ and strictly decreasing on $[0,\varphi(0)]$.

(ii) $\forall t \in \mathbf{I} : \varphi^{[-1]}(\varphi(t)) = t$

(iii) $\forall t \in [0,\infty] : \varphi(\varphi^{[-1]}(t)) = \min(t, \varphi(0))$

(iv) $\varphi(0) = \infty \Rightarrow \varphi^{[-1]} = \varphi^{-1}$

Proof. (i) These properties are an immediate consequence of the definition of $\varphi^{[-1]}$ and properties of an inverse function.

(ii) Let $t \in \mathbf{I}$ since φ is strictly decreasing $\varphi(t) \leq \varphi(0)$. Therefore per definition of $\varphi^{[-1]}$ it holds

$$\varphi^{[-1]}(\varphi(t)) = t.$$

(iii) Let $t \geq 0$. If $t \leq \varphi(0)$, we have

$$\varphi(\varphi^{[-1]}(t)) = \varphi(\varphi^{-1}(t)) = t.$$

If $t > \varphi(0)$, we get

$$\varphi(\varphi^{[-1]}(t)) = \varphi(0).$$

The claim follows.

(iv) Follows directly from the definition of $\varphi^{[-1]}$.

■

In order to show that a function C defined by (1.1) is indeed a copula, we will need the two following Lemmata.

Lemma 1.3 Let $\varphi : \mathbf{I} \to [0, \infty]$ continuous, strictly decreasing such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$$

Then C satisfies the boundary conditions for a copula that is for every $u, v \in \mathbf{I}$

$$C(u, 0) = C(0, v) = 0$$

and

$$C(u, 1) = u, \quad C(1, v) = v.$$

Proof. Let $u \in \mathbf{I}$. By definition of $\varphi^{[-1]}$ it follows that

$$C(u, 0) = \varphi^{[-1]}(\underbrace{\varphi(u) + \varphi(0)}_{\geq \varphi(0)}) = 0$$

and

$$C(u, 1) = \varphi^{[-1]}(\varphi(u) + \varphi(1)) = \varphi^{[-1]}(\varphi(u)) = u$$

Since C is obviously symmetric, the proof for the other component remains the same. ■

Lemma 1.4 Let $\varphi : \mathbf{I} \to [0, \infty]$ continuous, strictly decreasing such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)).$$

Then C is 2-increasing if and only if for all $v \in \mathbf{I}$:

$$u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1.$$

Proof. "$\Rightarrow$" Let $u_1, u_2, v \in \mathbf{I}$ with $u_1 \leq u_2$. Now using Lemma 1.3

$$C(u_2, v) - C(u_1, v) \leq u_2 - u_1$$

is equivalent to

$$C(u_2, 1) - C(u_1, 1) - C(u_2, v) + C(u_1, v) \geq 0.$$

So we have $V_C([u_1, u_2] \times [v, 1]) \geq 0$, which is true if C is 2-increasing.

"$\Leftarrow$" Now assume that for all $v \in \mathbf{I}$:

$$u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1.$$

Let $u_1, u_2, v_1, v_2 \in \mathbf{I}$ with $v_1 \leq v_2$ and $u_1 \leq u_2$. By Lemma 1.2.

$$C(0, v_2) = 0 \leq v_1 \leq v_2 = C(1, v_2).$$

So by using the intermediate value theorem on the continuous function $s \mapsto C(s, v_2), s \in \mathbf{I}$, there exists a t in $\mathbf{I}$ with $C(t, v_2) = v_1$, or equivalently $\varphi(t) + \varphi(v_2) = \varphi(v_1)$. Using this property we get

$$\begin{aligned} C(u_2, v_1) - C(u_1, v_1) = & \quad \varphi^{[-1]}(\varphi(u_2) + \varphi(v_2) + \varphi(t)) - \\ & \quad \varphi^{[-1]}(\varphi(u_1) + \varphi(v_2) + \varphi(t)) \\ = & \quad C(C(u_2, v_2), t) - C(C(u_1, v_2), t) \\ \leq & \quad C(u_2, v_2) - C(u_1, v_2). \end{aligned}$$

Where we have used our assumption in the last inequality, since $C(u_1, v_2) \leq C(u_2, v_2)$. Now

$$C(u_2, v_1) - C(u_1, v_1) \leq C(u_2, v_2) - C(u_1, v_2)$$

is equivalent to $V_C([u_1, u_2] \times [v_1, v_2]) \geq 0$. This completes the proof. ■

The following Theorem is an important result of this chapter, since it gives a necessary and sufficient condition under which circumstances C defined by (1.1) is a copula. For the proof we will need Lemma 1.3 and Lemma 1.4.

Theorem 1.5 Let $\varphi : \mathbf{I} \to [0, \infty]$ be continuous, strictly decreasing and such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined as before. Then C is a copula if and only if φ is convex.

Proof. By the preceding Lemmata it is enough to show that

$$\forall v \in \mathbf{I} : (u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1) \Leftrightarrow \varphi \text{ convex}.$$

Now since φ is strictly decreasing, φ is convex if and only if $\varphi^{[-1]}$ is convex. We will therefore show

$$\forall v \in \mathbf{I} : (u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1) \Leftrightarrow \varphi^{[-1]} \text{ convex}.$$

The statement on the left can be written as

$$u_1 + \varphi^{[-1]}(\varphi(u_2) + \varphi(v)) \leq u_2 + \varphi^{[-1]}(\varphi(u_1) + \varphi(v))$$

if $u_1 \leq u_2$. Now defining $a := \varphi(u_1)$,$b := \varphi(u_2)$ and $c := \varphi(v)$ this can be seen as

$$\varphi^{[-1]}(a) + \varphi^{[-1]}(b + c) \leq \varphi^{[-1]}(b) + \varphi^{[-1]}(a + c). \tag{1.2}$$

"$\Rightarrow$" Now assume that (1.2) holds. For $s, t \in [0, \infty]$ with $s < t$ set $a = (s+t)/2$, $b = s$, $c = (t-s)/2$. So (1.2) yields

$$\varphi^{[-1]}\left(\frac{s+t}{2}\right) + \varphi^{[-1]}\left(\frac{t+s}{2}\right) \leq \varphi^{[-1]}(s) + \varphi^{[-1]}(t),$$

which is equivalent to

$$\varphi^{[-1]}\left(\frac{s+t}{2}\right) \leq \frac{\varphi^{[-1]}(s) + \varphi^{[-1]}(t)}{2}.$$

Therefore $\varphi^{[-1]}$ is midpoint-convex and since $\varphi^{[-1]}$ is Lebesgue measurable as a continuous function, it follows that $\varphi^{[-1]}$ is convex.

"$\Leftarrow$" Let $\varphi^{[-1]}$ be a convex function. Furthermore, let $a, b, c \in \mathbf{I}$, with $a \geq b$ and $c \geq 0$. Define $\gamma := (a-b)/(a-b+c) \in [0,1]$. Note that $a = (1-\gamma)b + \gamma(a+c)$ and $b + c = \gamma b + (1-\gamma)(a+c)$. Now using the convexity of $\varphi^{[-1]}$ we get

$$\varphi^{[-1]}(a) \leq (1-\gamma)\varphi^{[-1]}(b) + \gamma\varphi^{[-1]}(a+c)$$

and

$$\varphi^{[-1]}(b+c) \leq \gamma\varphi^{[-1]}(b) + (1-\gamma)\varphi^{[-1]}(a+c).$$

Adding these inequalities yields

$$\varphi^{[-1]}(a) + \varphi^{[-1]}(b+c) \leq \varphi^{[-1]}(b) + \varphi^{[-1]}(a+c)$$

and this is exactly (1.2). The claim follows. ∎

Now we are finally ready to define Archimedean copulas as follows.

Definition 1.6 Let $\varphi : \mathbf{I} \to [0, \infty]$ be continuous, strictly decreasing, convex and such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined by

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)).$$

By Theorem 1.5 C is a copula, called an Archimedean copula. φ is called a generator of C. If $\varphi(0) = \infty$, we say φ is a strict generator and C is a strict Archimedean copula.

In the following example we will proof that two well-known copulas we have already discussed are Archimedean copulas. We will refer to this example quite a lot throughout this paper.

Example 1.7 (a) Let $\varphi(t) = -\log(t)$, $t \in [0,1]$. Since $\varphi(0) = \infty$, by Lemma 1.2 (iv) it follows that $\varphi^{[-1]}(t) = \varphi^{-1}(t) = \exp(-t)$. So

$$C(u,v) = \exp(-[(-\log u) + (-\log v)]) = \exp(\log(uv)) = uv = \Pi(u,v).$$

(b) Let $\varphi(t) = 1 - t$, $t \in [0,1]$. In this case it holds

$$\varphi^{[-1]}(t) = \begin{cases} 1-t, & 0 \leq t \leq 1 \\ 0, & 1 \leq t \leq \infty \end{cases}.$$

Therefore

$$C(u,v) = \varphi^{[-1]}(2-u-v) = \begin{cases} u+v-1, & 0 \leq 2-u-v \leq 1 \\ 0, & 1 \leq 2-u-v \end{cases} = \max(u+v-1, 0).$$

So we see $C = W$, where W is the lower Frechet-Hoeffding bound for copulas.

2 One-parameter families

The great benefits of the class of Archimedean copulas were already hinted at in the last chapter. First note that it is really easy to construct Archimedean copulas. By Theorem 1.5 we only have to find a suitable generator function φ to construct such a copula. In addition to that the class of Archimedean copulas consists of a wide variety of different families and has therefore a lot of applications, for example in statistics. We will give two example one-parameter families of Archimedean copulas here.

Example 2.1 First we will take a look at the Gumbel-Hougaard family given by

$$C_\theta(u,v) = \exp(-[(-\log u)^\theta] + (-\log(v)^\theta)]^{1/\theta})$$

for $\theta \in [1,\infty)$. The family of generators is given by

$$\varphi_\theta(t) = (-\log t)^\theta.$$

Special cases in this family are $C_1 = \Pi$ and $C_\infty = M$, where $M(u,v) = \min(u,v), (u,v) \in \mathbf{I}^2$.

Example 2.2 The second example of an Archimedean family is given by

$$C_\theta(u,v) = \max(\theta uv + (1-\theta)(u+v-1), 0)$$

for $\theta \in (0,1]$. The family of generators is given by

$$\varphi_\theta(t) = -\log(\theta t + (1-\theta)).$$

Special cases in this family are $C_0 = W$ and $C_1 = \Pi$. In Figure 2.1 we can see a scatterplot of 500 samples of this family for different values of θ. Interesting is that we can already see as we get closer to 1 with θ, how our plot more and more resembles Π.

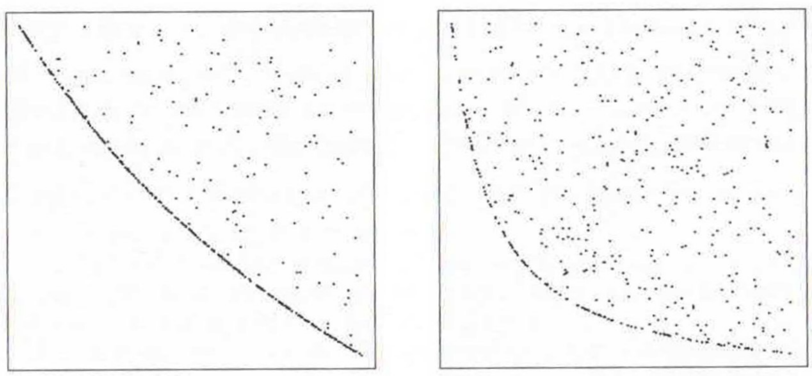

Figure 2.1: Scatterplots, $\theta = 0.4$ (left) and $\theta = 0.9$ (right)

Note that the special cases in these families are taken for values that are not necessarily in the parameter interval, for example C_0 in Example 2.2. We conclude that some limit has to be calculated there. How one can calculate these limiting cases of Archimedean copulas is discussed in the fourth chapter.

3 Fundamental properties

For simplicity let Ω denote the set of continuous strictly decreasing convex functions $\varphi : \mathbf{I} \to [0, \infty]$ with $\varphi(1) = 0$. So Ω is the set of generator functions for Archimedean copulas. The next theorem and the following corollaries are used in chapter 4 to prove a theorem about limiting cases of Archimedean copulas.

Theorem 3.1 Let C be an Archimedean copula generated by $\varphi \in \Omega$. Let $K_C(t)$ denote the C-measure of the set

$$\{(u,v) \in \mathbf{I}^2 | C(u,v) \le t\} = \{(u,v) \in \mathbf{I}^2 | \varphi(u) + \varphi(v) \ge \varphi(t)\}.$$

Then for any t in $\mathbf{I}$

$$K_C(t) = t - \frac{\varphi(t)}{\varphi'(t+)},$$

where $\varphi'(t+)$ denotes the right-sided derivative of φ at t.

Proof. First note that $\varphi'(t+)$ exists, since φ is convex. Let t be in $(0,1)$, and set $w = \varphi(t)$. Let $n \in \mathbb{N}$. Let $W := \{0, \frac{w}{n}, \dots, \frac{wn}{n}\}$ be a partition of $[0, w]$ and $T := \{t = t_0, \dots, t_n = 1\}$ be a partition of $[t, 1]$ with

$$t_{n-k} = \varphi^{[-1]}(\frac{kw}{n}), \quad k = 0, 1, \dots, n.$$

It follows that

$$C(t_j, t_k) = \varphi^{[-1]}(\varphi(t_j) + \varphi(t_k)) = \varphi^{[-1]}(w + \frac{n-j-k}{n} w),$$

especially $C(t_j, t_{n-j}) = t$.

Denote $[t_{k-1}, t_k] \times [0, t_{n-k+1}]$ by R_k, and let $S_n = \cup_{k=1}^n R_k$.

Note that by using the convexity of $\varphi^{[-1]}$

$$0 \le t_1 - t_0 \le \dots \le t_n - t_{n-1}$$

and $\lim_{n\to\infty} t_n - t_{n-1} = \lim_{n\to\infty} 1 - \varphi^{[-1]}\left(\frac{w}{n}\right) = 0$. So the mesh of our partition converges to 0 as $n \to \infty$. Therefore we have, by the construction of the C-measure, $K_C(t)$ is the sum of the C-measure of $[0, t] \times \mathbf{I}$ and $\lim_{n\to\infty} V_C(S_n)$.

Now we calculate for each k

$$\begin{aligned} V_C(R_k) &= C(t_k, t_{n-k+1}) - C(t_k, 0) - C(t_{k-1}, t_{n-k+1}) + C(t_{k-1}, 0) \\ &= C(t_k, t_{n-k+1}) - t \\ &= \varphi^{[-1]}(w - \frac{w}{n}) - \varphi^{[-1]}(w) \end{aligned}$$

and since R_k and R_l are disjoint for $k \ne l$ (apart from a set with C-measure 0)

$$V_C(S_n) = \sum_{k=1}^n V_C(R_k) = nV_C(R_1) = -w \left[\frac{\varphi^{[-1]}(w) - \varphi^{[-1]}(w - w/n)}{w/n} \right].$$

So using the rule for the derivative of an inverse function

$$\lim_{n\to\infty} V_C(S_n) = -\frac{w}{(\varphi^{[-1]})'(w-)} = -\frac{w}{\varphi'(t+)}.$$

Finally

$$K_C(t) = V_C([0,t] \times \mathbf{I}) + \lim_{n\to\infty} V_C(S_n) = t - \frac{\varphi(t)}{\varphi'(t+)}.$$

■

Corollary 3.2 Let C be an Archimedean copula generated by $\varphi \in \Omega$. Let $K'_C(s,t)$ denote the C-measure of the set

$$\{(u,v) \in \mathbf{I}^2 | u \le s, C(u,v) \le t\}$$

Then for any $(s,t) \in \mathbf{I}^2$

$$K'_C(s,t) = \begin{cases} s, & s \le t \\ t - \frac{\varphi(t)-\varphi(s)}{\varphi'(t^+)} & s > t \end{cases}$$

The next corollary gives a probabilistic interpretation of the results we proved in Theorem 3.1 and Corollary 3.2.

Corollary 3.3 Let U and V be uniform (0,1) random variables with joint distribution function C generated by $\varphi \in \Omega$. Then the function K_C is the distribution function of $C(U,V)$. Furthermore, the function K'_C is the joint distribution function of U and $C(U,V)$.

Proof. Note that the C-Measure of a set $A \subset \mathbf{I}^2$ is the probability that two uniform $(0,1)$ random variables, with joint distribution function C are inside this set. So by the choice of (U,V) and the definition of $K_C(t)$:

$$K_C(t) = P((U,V) \in \{(u,v) \in \mathbf{I}^2 | C(u,v) \le t\}) = P(C(U,V) \le t),$$

which shows that K_C is the distribution function of $C(U,V)$. For K'_C the claim follows similarly. ■

4 Order and limiting cases

In this chapter we will portrait some nice properties, members of the class of Archimedean copulas possess. We will start with the order.

Definition 4.1 Let C_1 and C_2 be copulas, we say C_1 is smaller than C_2 (or C_2 is larger than C_1), and write $C_1 \prec C_2$ (or $C_2 \succ C_1$) if $C_1(u,v) \leq C_2(u,v)$ for all u, v in $\mathbf{I}$.

We say a family $\{C_\theta\}$ of copulas is positively ordered if

$$\alpha \leq \beta \Rightarrow C_\alpha \prec C_\beta.$$

The family is negatively ordered if

$$\alpha \leq \beta \Rightarrow C_\alpha \succ C_\beta.$$

Example 4.2 Consider this one-parameter family of Archimedean copulas

$$C_\theta(u,v) = \theta / \log(e^{\theta/u} + e^{\theta/v} - e^\theta),$$

generated by

$$\varphi_\theta(t) = e^{\theta/t} - e^\theta$$

for $\theta \in (0, \infty)$.
Now let $\theta_1, \theta_2 \in (0, \infty)$, $\theta_1 \leq \theta_2$. Is there a relation between

$$\frac{\theta_1}{\log(e^{\theta_1/u} + e^{\theta_1/v} - e_1^\theta)} \text{ and } \frac{\theta_2}{\log(e^{\theta_2/u} + e^{\theta_2/v} - e_2^\theta)}?$$

As this example illustrates, using the definition to check wether a family of copulas has an order, can be quite difficult. For Archimedean copulas we are in a better situation and can use a different criteria to determine if a family is ordered. For that we will need the definition of subadditivity.

Definition 4.3 A function $f : [0, \infty) \to \mathbb{R}$ is subadditive if for all $x, y \in [0, \infty)$

$$f(x+y) \leq f(x) + f(y).$$

Theorem 4.4 Let C_1 and C_2 be Archimedean copulas generated by φ_1 and φ_2 in Ω. Then $C_1 \prec C_2$ if and only if $\varphi_1 \circ \varphi_2^{[-1]}$ is subadditive.

Proof. Let $f = \varphi_1 \circ \varphi_2^{[-1]}$. f is continuous, nondecreasing, and $f(0) = 0$. Per definitionem, $C_1 \prec C_2$ if and only if for all u, v in $\mathbf{I}$,

$$\varphi_1^{[-1]}(\varphi_1(u) + \varphi_1(v)) \leq \varphi_2^{[-1]}(\varphi_2(u) + \varphi_2(v)).$$

Let $x = \varphi_2(u)$ and $y = \varphi_2(v)$, then the above is equivalent to

$$\varphi_1^{[-1]}(f(x) + f(y)) \leq \varphi_2^{[-1]}(x+y) \tag{4.1}$$

for all x, y in $[0, \varphi_2(0)]$. In addition if $x > \varphi_2(0)$ or $y > \varphi_2(0)$, then (4.1) reduces to $0 \leq 0$. This can be seen as follows. Let for example $x > \varphi_2(0)$, the same reasoning applies if $y > \varphi_2(0)$.

By definition of the pseudo-inverse

$$\varphi_2^{[-1]}(x+y) = 0.$$

Additionally since $\varphi_2^{[-1]}(x) = 0$, we get that $f(x) = \varphi_1(0)$ and therefore

$$\varphi_1^{[-1]}(f(x)+f(y)) = \varphi_1^{[-1]}(\underbrace{\varphi_1(0)+f(y)}_{\geq \varphi_1(0)}) = 0.$$

"$\Rightarrow$" Now let $C_1 \prec C_2$. So we know (4.1) holds for $x, y \in [0, \infty)$. Applying φ_1 to both sides yields

$$\varphi_1(\varphi_1^{[-1]}(f(x)+f(y))) \geq \varphi_1(\varphi_2^{[-1]}(x+y)).$$

Now since $\varphi(\varphi^{[-1]})(t) \leq t$, for $t \in [0, \infty]$,

$$f(x+y) \leq f(x) + f(y),$$

so f is subadditive.
"$\Leftarrow$" Conversely let f be subadditive we can apply $\varphi_1^{[-1]}$ to

$$f(x+y) \leq f(x) + f(y)$$

and we get

$$\varphi_1^{[-1]}(f(x+y)) \geq \varphi_1^{[-1]}(f(x)+f(y)).$$

By definition of f

$$\varphi_1^{[-1]}(f(x)+f(y)) \leq \varphi_2^{[-1]}(x+y)$$

and this is the claim. ■

In the next theorem we will see under which circumstances the limit of a family of Archimedean copulas is an Archimedean copula. This theorem was used to calculate the limiting cases in Examples 2.1 and 2.2.

Theorem 4.5 Let $\{C_\theta | \theta \in \Theta\}$ be a family of Archimedean copulas with differentiable generators φ_θ in Ω. Then $C = \lim C_\theta$ (the limit is understood as a pointwise limit) is an Archimedean copula if and only if there exists a function φ in Ω such that for all s, t in $(0, 1)$:

$$\lim \frac{\varphi_\theta(s)}{\varphi_\theta'(t)} = \frac{\varphi(s)}{\varphi'(t)},$$

where lim denotes the appropriate one-sided limit as θ approaches an end point of the parameter interval. The generator of C is φ.

Proof. Let (U_θ, V_θ) be uniform (0,1) random variables with joint distribution function C_θ, let K_θ denote the distribution function of $C_\theta(U_\theta, V_\theta)$ and let K_θ' denote the joint distribution function of U_θ and $C_\theta(U_\theta, V_\theta)$. By Corollaries 3.2. and 3.3. we get

$$K_\theta'(s,t) = t - \frac{\varphi_\theta(t)}{\varphi_\theta'(t)} + \frac{\varphi_\theta(s)}{\varphi_\theta'(t)}$$

for $0 < t < s < 1$ and

$$K_\theta(t) = t - \frac{\varphi_\theta(t)}{\varphi_\theta'(t)}$$

for all t in $\mathbf{I}$. Now let (U, V) be uniform (0,1) random variables with joint distribution function C, let K be the distribution function of $C(U, V)$ and let K' denote the joint distribution function of U and $C(U, V)$.

Assume $C = \lim C_\theta$ is Archimedean with generator φ. So

$$\lim t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)} = \lim K_\theta(t) = K(t) = t - \frac{\varphi(t)}{\varphi'(t)} \tag{4.2}$$

for $t \in \mathbf{I}$. Note that the equality $\lim K_\theta(t) = K(t)$ is a consequence of the definition of the C-volume and construction of the C-measure. This proves the claim for $s = t$.
For $0 < t < s < 1$. It now holds similarly that

$$\lim t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)} + \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \lim K'_\theta(s,t) = K'(s,t) = t - \frac{\varphi(t)}{\varphi'(t)} + \frac{\varphi(s)}{\varphi\prime(t)}$$

and with (4.2) we get

$$\lim \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \frac{\varphi(s)}{\varphi'(t)}.$$

Conversely assume that for all s, t in $(0,1)$:

$$\lim \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \frac{\varphi(s)}{\varphi'(t)}.$$

If we denote $c_\theta := \frac{\varphi'(t_0)}{\varphi'_\theta(t_0)} > 0$ for a fixed $t_0 \in (0,1)$, we get that for all $s \in (0,1]$, $\lim c_\theta \varphi_\theta(s) = \varphi(s)$.

Claim: $\lim \varphi_\theta^{[-1]}\left(\frac{\cdot}{c_\theta}\right) = \varphi^{[-1]}(\cdot)$.
For that let $\varepsilon > 0$, and $s \in (0, \varphi(0))$. Since $\varphi^{[-1]}$ is continuous, there exists $\delta > 0$, such that

$$\max\{|\varphi^{[-1]}(s+\delta) - \varphi^{[-1]}(s)|, |\varphi^{[-1]}(s-\delta) - \varphi^{[-1]}(s)|\} < \varepsilon,$$

and $s + \delta, s - \delta \in (0, \varphi(0))$.
Now since $c_\theta \varphi_\theta \to \varphi$ pointwise, if θ is chosen close enough to an endpoint of the parameter interval it holds

$$\begin{aligned}
&\max\{|c_\theta\varphi_\theta(\varphi^{[-1]}(s+\delta)) - \varphi(\varphi^{[-1]}(s+\delta))|, |c_\theta\varphi_\theta(\varphi^{[-1]}(s-\delta)) - \varphi(\varphi^{[-1]}(s-\delta))|\} \\
&= \max\{|c_\theta\varphi_\theta(\varphi^{[-1]}(s+\delta)) - (s+\delta)|, |c_\theta\varphi_\theta(\varphi^{[-1]}(s-\delta)) - (s-\delta)|\} < \delta.
\end{aligned}$$

So it has to hold

$$c_\theta\varphi_\theta(\varphi^{[-1]}(s-\delta)) < s < c_\theta\varphi_\theta(\varphi^{[-1]}(s+\delta)).$$

If we apply $\varphi_\theta^{[-1]}(\cdot/c_\theta)$ to the above and note that $\varphi_\theta^{[-1]}(\cdot/c_\theta)$ is strictly decreasing (c_θ is positive), we end up with

$$\varphi^{[-1]}(s+\delta) < \varphi_\theta^{[-1]}\left(\frac{s}{c_\theta}\right) < \varphi^{[-1]}(s-\delta).$$

But by the choice of δ, we get now that $\varphi^{[-1]}(s+\delta)$,$\varphi^{[-1]}(s-\delta) \in [\varphi^{[-1]}(s) - \varepsilon, \varphi^{[-1]}(s) + \varepsilon]$. So

$$|\varphi^{[-1]}(s) - \varphi_\theta^{[-1]}\left(\frac{s}{c_\theta}\right)| < \varepsilon$$

and this is our claim. For $s = 0$ the claim also holds as a direct consequence of the definition of a generator. For $s = \varphi(0)$, we can use a similar argument using the continuity of φ and the definition of the pseudo-inverse.

We will show $C = \lim C_\theta$ is Archimedean with generator φ. For that using the claim, the continuity of $\varphi^{[-1]}$ and $\lim c_\theta \varphi_\theta = \varphi$, we have

$$\lim_{\theta} \lim_{\theta_1} \varphi_\theta^{[-1]}[\frac{c_{\theta_1}}{c_\theta}(\varphi_{\theta_1}(u) + \varphi_{\theta_1}(v))] = \lim_{\theta} \varphi_\theta^{[-1]} \left[\frac{\varphi(u) + \varphi(v)}{c_\theta} \right] = \varphi^{[-1]}[\varphi(u) + \varphi(v)],$$

as well as

$$\lim_{\theta_1} \lim_{\theta} \varphi_\theta^{[-1]}[\frac{c_{\theta_1}}{c_\theta}(\varphi_{\theta_1}(u) + \varphi_{\theta_1}(v))] = \lim_{\theta_1} \varphi^{[-1]}[c_{\theta_1}(\varphi_{\theta_1}(u) + \varphi_{\theta_1}(v))] = \varphi^{[-1]}[\varphi(u) + \varphi(v)].$$

We can therefore set $\theta = \theta_1$, hence

$$\lim_{\theta} \varphi_\theta^{[-1]}[\varphi_\theta(u) + \varphi_\theta(v)] = \lim_{\theta} \lim_{\theta_1} \varphi_\theta^{[-1]}[\frac{c_{\theta_1}}{c_\theta}(\varphi_{\theta_1}(u) + \varphi_{\theta_1}(v))] = \varphi^{[-1]}[\varphi(u) + \varphi(v)]$$

for fixed $u, v \in \mathbf{I}$ and this completes the proof. ■

We can now calculate the two limiting cases in Example 2.2.

Example 4.6

- Let $\varphi_\theta(t) = -\log(\theta t + (1 - \theta))$, $\theta \in (0, 1]$. Using L'Hospital:

  $$\lim_{\theta \to 0+} \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \lim_{\theta \to 0+} \frac{\log(\theta s + (1 - \theta))}{\theta/(\theta t + (1 - \theta))} = s - 1.$$

  Now define $\varphi(s) = 1 - s, s \in \mathbf{I}$. So it is easy to see that

  $$\lim_{\theta \to 0+} \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \frac{\varphi(s)}{\varphi'(t)}.$$

  By Theorem 4.5 it follows that C_0 is Archimedean with generator φ, so using Example 1.7 (b) we see that $C_0 = W$.

- Using the same family of generators, we have

  $$\lim_{\theta \to 1-} \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \lim_{\theta \to 1-} \frac{\log(\theta s + (1 - \theta))}{\theta/(\theta t + (1 - \theta))} = t \log s.$$

  Now define $\varphi(s) = -\log(s), s \in \mathbf{I}$. So by the above

  $$\lim_{\theta \to 1-} \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \frac{\varphi(s)}{\varphi'(t)}.$$

  Again we see that C_1 is Archimedean with generator φ, so using Example 1.7 (a) we see that $C_1 = \Pi$.

5 Two-parameter families

In this chapter we will give one way of constructing two-parameter families. The main idea is to compose generators with the power function $t \mapsto t^\theta$, $\theta > 0$. One benefit of this approach is that we can easily construct a two-parameter family of Archimedean copulas, by only knowing one generator.

Theorem 5.1 Let $\varphi \in \Omega$, let $\alpha, \beta > 0$ and define

$$\varphi_{\alpha,1}(t) = \varphi(t^\alpha) \qquad \varphi_{1,\beta}(t) = [\varphi(t)]^\beta.$$

If $\beta \geq 1$, then $\varphi_{1,\beta} \in \Omega$. If α is in $(0,1]$, then $\varphi_{\alpha,1} \in \Omega$.
If φ is twice differentiable and $t\varphi'(t)$ is nondecreasing on $(0,1)$, then $\varphi_{\alpha,1}$ is an element of Ω for all $\alpha > 0$.

Proof. We only show that for $\beta \geq 1$ it holds that $\varphi_{1,\beta} \in \Omega$. The rest of the claims follow similarly. So let $\beta \geq 1$. $\varphi_{1,\beta}$ is continuous, since the power function and φ are continuous. To show the convexity of $\varphi_{1,\beta}$, first note that the function $h : [0,\infty] \to \mathbb{R}, h(x) = x^\beta$ is convex. We can see that by calculating $h'(x) = \beta x^{\beta-1} \geq 0$, so h is convex. Now let $t_1, t_2 \in \mathbf{I}$ and $\lambda \in [0,1]$. We get

$$(\varphi(\lambda t_1 + (1-\lambda)t_2))^\beta \leq (\lambda\varphi(t_1) + (1-\lambda)\varphi(t_2))^\beta \leq \lambda\varphi(t_1)^\beta + (1-\lambda)\varphi(t_2)^\beta,$$

where we used that h is nondecreasing, and φ is convex in the first inequality and the convexity of h in the second one. We show now that $\varphi_{1,\beta}$ is striclty decreasing. Let $t_1, t_2 \in \mathbf{I}$, $t_1 < t_2$, so

$$\varphi(t_1)^\beta > \varphi(t_2)^\beta.$$

We used that φ is strictly decreasing and h is strictly increasing. Finally

$$\varphi_{1,\beta}(1) = \varphi(1)^\beta = 0^\beta = 0,$$

which completes the proof. ■

To construct a two-parameter family of copulas we can now define

$$\varphi_{\alpha,\beta}(t) = [\varphi(t^\alpha)]^\beta$$

and note that $\varphi_{\alpha,\beta} \in \Omega$ if we choose α and β as in Theorem 5.1.

Example 5.2 Let $\varphi(t) = 1 - t$ and using our approach we define $\varphi_{\alpha,\beta}(t) = (1-t^\alpha)^\beta$ for $\alpha \in (0,1]$, $\beta \geq 1$. This generates

$$C_{\alpha,\beta}(u,v) = \max\left(\left[1 - ((1-u^\alpha)^\beta + (1-v^\alpha)^\beta)^{1/\beta}\right]^{1/\alpha}, 0\right).$$

Special cases in this family are for example $C_{1,1} = W$, $C_{0,1} = \Pi$ and $C_{\alpha,\infty} = M$.

References

[1] Nelson, R. B. *An Introduction to Copulas*. 2006.

[2] Hofert, M., Kojadinovic, I., Mächler, M., Yan, J. *Elements of Copula Modeling with R*. 2018.