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Geometric Methods of Constructing Copulas

Tim Donhauser | Februar 2021 | Seminar: Copulas and their Applications

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Recap: Singular Copulas

Definition

A copula C is said to be *singular* if it has support $S(C)$ with Lebesgue measure 0, i.e.

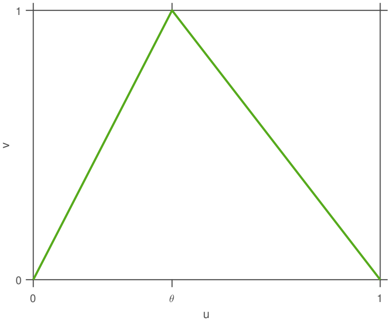
$$\lambda\big(S(C)\big) = 0,$$

where $S(C)$ is defined by

$$S(C) := \{A \subset \mathbb{I}^2 : A \text{ open and } V_C(A) = 0\}^c$$

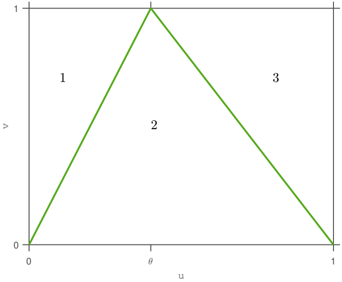
Example 1:

Let $\theta \in [0, 1]$ and suppose that the support of the desired copula C_θ is given by the lines connecting the points $(0, 0), (\theta, 1)$ and $(\theta, 1), (1, 0)$:



Three Cases to Consider:

1. $u \leq \theta v$
2. $u > \theta v$ and $u < 1 - (1 - v)\theta$
3. $u \geq 1 - (1 - v)\theta$



1. $u \leq \theta v$:

$$
\begin{aligned}
C_\theta(u, v) &= V_{C_\theta}([0, u] \times [0, v]) \\
&= V_{C_\theta}([0, u] \times [0, v]) + \underbrace{V_{C_\theta}([0, u] \times [v, 1])}_{=0} \\
&= V_{C_\theta}([0, u] \times [0, 1]) \\
&= C_\theta(u, 1) \\
&= u
\end{aligned}
$$

2. $u > \theta v$ and $u < 1 - (1 - v)\theta$:

$$
\begin{aligned}
C_\theta(u, v) &= V_{C_\theta}([0, u] \times [0, v]) \\
&= V_{C_\theta}([0, \theta v] \times [0, v]) + \underbrace{V_{C_\theta}([\theta v, u] \times [0, v])}_{=0} \\
&= C_\theta(\theta v, v) \\
&= V_{C_\theta}([0, \theta v] \times [0, v]) \\
&= V_{C_\theta}([0, \theta v] \times [0, v]) + \underbrace{V_{C_\theta}([0, \theta v] \times [v, 1])}_{=0} \\
&= V_{C_\theta}([0, \theta v] \times [0, 1]) \\
&= C_\theta(\theta v, 1) \\
&= \theta v
\end{aligned}
$$

3. $u \geq 1 - (1 - v)\theta$:

Here, it holds that

$$V_{C_\theta}([u, 1] \times [v, 1]) = 0$$

and

$$V_{C_\theta}([u, 1] \times [v, 1]) = C_\theta(u, v) - u - v + 1,$$

which results in

$$C_\theta(u, v) = u + v - 1.$$

The resulting copula C_θ is given by:

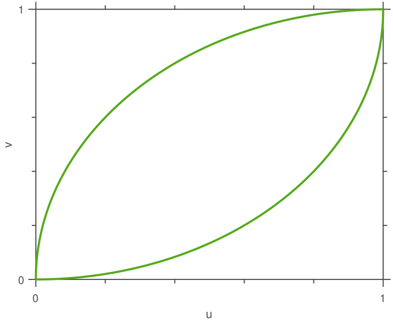
$$C_\theta(u,v) = \begin{cases} u & \text{, if } 0 \leq u \leq \theta v \leq \theta \\ \theta v & \text{, if } 0 \leq \theta v < u < 1-(1-\theta)v \\ u+v-1 & \text{, if } \theta \leq 1-(1-\theta)v \leq u \leq 1 \end{cases}$$

Example 2: Constructing a Symmteric Copula

Let the prescribed support be given by the set

$$S(C) := \{(u, v) \in \mathbb{I}^2 : u^2 + v^2 = 2u\} \cup \{(u, v) \in \mathbb{I}^2 : u^2 + v^2 = 2v\},$$

whose graph looks like this:



1. $u^2 + v^2 > 2\min(u, v)$:

Since for $u^2 + v^2 > 2u$ it must hold that

$$V_C([0, u] \times [v, 1]) = u - C(u, v) \overset{!}{=} 0,$$

we have

$$C(u, v) = u$$

and analogue for $u^2 + v^2 > 2v$:

$$C(u, v) = v.$$

With the Fréchet-Hoeffding upper boundary we get

$$C(u, v) \leq M(u, v) := \min(u, v) \quad \Rightarrow \quad C(u, v) = M(u, v).$$

2. $u \leq v$ and $u^2 + v^2 \leq 2u$:

Here, it must hold that

$$V_C([u,v] \times [u,v]) = 0 \quad \Leftrightarrow \quad C(u,v) + C(v,u) = C(u,u) + C(v,v)$$

and since we have symmetry

$$2C(u,v) = C(u,u) + C(v,v).$$

Analogue for $v \leq u$ and $u^2 + v^2 \leq 2v$.

Considering $u^2 + v^2 = 2u$ does the Trick:

By continuity reasons the two previous slides give

$$u = C(u, v) = \frac{1}{2}\big(C(u, u) + C(v, v)\big),$$

which is equivalent to

$$C(u, u) + C(v, v) = 2u = u^2 + v^2.$$

This can be solved by

$$C(u, u) = u^2$$

for any $u \in \mathbb{I}$, resulting in

$$C(u, v) = \min\left(u, v, \frac{u^2 + v^2}{2}\right), \quad \forall (u, v) \in \mathbb{I}^2.$$

Definition: Ordinal Sums

Let K be a (possibly) finite index set, $\{J_k\}_{k\in K}$ a partition of $\mathbb{I}$ with $J_k = [a_k, b_k]$, for $k \in K$, and $\{C_k\}_{k\in K}$ a collection of copulas. Then the *ordinal sum* of $\{C_k\}_{k\in K}$ w.r.t. $\{J_k\}_{k\in K}$ is defined by

$$C(u, v) = a_k + (b_k - a_k)C_k\left(\frac{u - a_k}{b_k - a_k}, \frac{v - a_k}{b_k - a_k}\right), \quad \text{if } (u, v) \in J_k^2$$

and

$$C(u, v) = M(u, v), \quad \text{if } (u, v) \notin J_k^2.$$

Theorem

Let C be a copula. Then C is an ordinal sum if and only if there exists a $t \in (0, 1)$ such that $C(t, t) = t$.

Proof.

"$\Rightarrow$" For any $k \in K$, take $t = a_k$ or $t = b_k$ to obtain

$$C(a_k, a_k) = a_k + (b_k - a_k)C_k(0, 0) = a_k$$

or $C(b_k, b_k) = b_k$, respectively.
"$\Leftarrow$" Assume $\exists t \in (0, 1)$ such that $C(t, t) = t$. Define for $(u, v) \in \mathbb{I}^2$

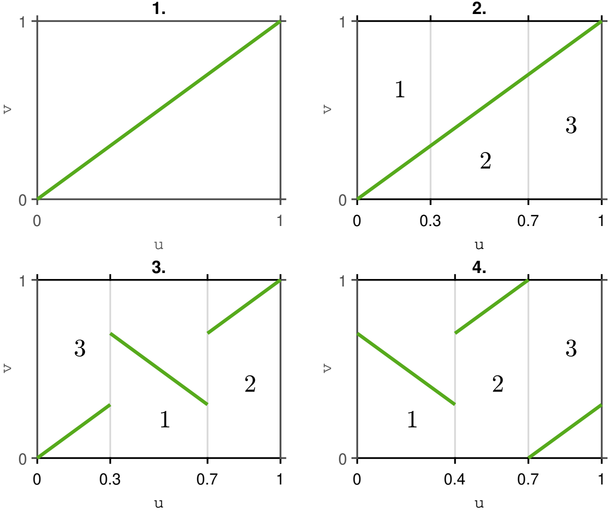
$$C_1(u, v) := \frac{C(tu, tv)}{t}$$

and

$$C_2(u, v) := \frac{C\big(t + (1 - t)u, t + (1 - t)v\big)}{1 - t}.$$

Then C_1 and C_2 are copulas and C is the ordinal sum of $\{C_1, C_2\}$ w.r.t $\{[0, t], [t, 1]\}$. □

Shuffles of M



Formal Construction

Let $n \in \mathbb{N}$, $\{J_i\}_{i=1,\dots,n}$ a partition of $\mathbb{I}$, π a permutation on $S_n = \{1, \dots, n\}$ (shuffling) and ω a function with $\omega : S_n \to \{-1, 1\}$ (flipping). The resulting shuffle of M is then denoted by

$$M(n, \{J_i\}_{i=1,\dots,n}, \pi, \omega).$$

If $\omega \equiv 1$, we call the resulting copula a *straight shuffle*, if $\omega \equiv -1$, we call it a *flipped shuffle*.
Note, that

$$W = M(1, [0, 1], \mathrm{id}, -1).$$

Properties of Shuffles of M (1)

Definition

Let X, Y be two random variables. Then X and Y are called *mutually completely dependent*, if there exists a bijective function ϕ such that $\mathbb{P}(X = \phi(Y)) = 1$

Let the copula of some random variables X, Y be given by a shuffle of M. Then X and Y are mutually completely dependent, since the support of any shuffle of M is the graph of a bijective function.

Properties of Shuffles of M (2)

The next two theorems are proven in the paper belonging to this talk.

Theorem

For any $\epsilon > 0$ and any copula C, there exists a shuffle of M, denoted by C_ϵ, such that

$$\sup_{u,v \in \mathbb{I}} |C_\epsilon(u, v) - C(u, v)| < \epsilon.$$

The following result allows us to narrow the Fréchet-Hoeffding bounds.

Properties of Shuffles of M (3)

Theorem

Let C be a copula and suppose $C(a,b) = \theta$, where $(a,b) \in \mathbb{I}^2$ and θ satisfies $\max(a+b-1,0) \leq \theta \leq min(a,b)$. Then

$$C_L(u,v) \leq C(u,v) \leq C_U(u,v)$$

where

$$C_U = M\big(4, \{[0,\theta],[\theta,a],[a,a+b-\theta],[a+b-\theta,1]\},(1,3,2,4),1\big)$$

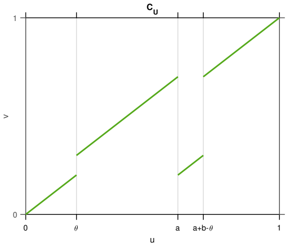
and

$$C_L = M\big(4, \{[0,a-\theta],[a-\theta,a],[a,1-b+\theta],[1-b+\theta,1]\},(4,2,3,1),-1\big)$$

Explicit Representation of C_U and C_L

$$C_U(u,v) = \min\left(u, v, \theta + (u-a)^+ + (v-b)^+\right)$$

$$C_L(u,v) = \max\left(0, u+v-1, \theta - (a-u)^+ + (b-v)^+\right)$$



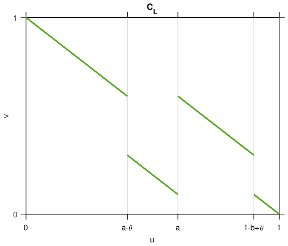


Figure: Support of C_U and C_L with $a = 0.6$, $b = 0.3$ and $\theta = 0.2$

Definition: Convex Sums

Let X be a continous random variable with distribution function F. Let C_x define a copula for any observation x of X. Then, the function defined by

$$C(u,v) = \int_{\mathbb{R}} C_x(u,v)dF(x)$$

is a called the *convex sum* of $\{C_x\}_{X=x}$ w.r.t. F, where F is called *mixing distribution*.
If F has a parameter α, we write

$$C_\alpha(u,v) = \int_{\mathbb{R}} C_x(u,v)dF_\alpha(x).$$

Convex Sums are Copulas:

For $u, v, u_1, u_2, v_1, v_2 \in \mathbb{I}$ with $u_1 \leq u_2$ and $v_1 \leq v_2$ we have:

(1)

$$C(0, v) = \int_{\mathbb{R}} C_x(0, v)dF(x) = \int_{\mathbb{R}} 0dF(x) = 0 = C(u, 0)$$

(2)

$$C(1, v) = \int_{\mathbb{R}} C_x(1, v)dF(x) = \int_{\mathbb{R}} vdF(x) = v \cdot 1 = v$$

(3)

$$V_C([u_1, u_2] \times [v_1, v_2]) = \int_{\mathbb{R}} V_{C_x}([u_1, u_2] \times [v_1, v_2])dF(x) \geq 0$$

Example

Let $\{C_x\}_{X=x}$ be a family of copulas defined by

$$C_x(u,v) = \begin{cases} M(u,v), \text{ if } |v-u| \geq x \\ W(u,v), \text{ if } |u+v-1| \geq 1-x \\ \frac{u+v-x}{2}, \text{ else} \end{cases}$$

for $x \in \mathbb{I}$. Let $F_\alpha(x) = x^\alpha$, $\alpha > 0$. Then C_α is given by

$$\begin{aligned} C_\alpha(u,v) &= \int_{\mathbb{I}} C_x(u,v) dF_\alpha(x) \\ &= \int_{\mathbb{I}} C_x(u,v) \alpha x^{\alpha-1} dx \end{aligned}$$

Example cont'd

(1) $$\int_{0}^{|v-u|} M(u,v)\alpha x^{\alpha-1}dx = M(u,v)|v-u|^{\alpha}$$

(2) $$\int_{1-|u+v-1|}^{1} W(u,v)\alpha x^{\alpha-1}dx = W(u,v)\left(1-(1-|u+v-1|)^{\alpha}\right)$$

(3) $$\int_{|v-u|}^{1-|u+v-1|} \frac{u+v-x}{2}\alpha x^{\alpha-1}dx = \frac{u+v}{2}\left((1-|u+v-1|)^{\alpha}-|v-u|^{\alpha}\right)$$
$$-\frac{\alpha}{2(\alpha+1)}\left((1-|u+v-1|)^{\alpha+1}-|v-u|^{\alpha+1}\right)$$

Four cases, one result

Assume $u \leq v$ and $u + v - 1 \geq 0$. Then

$$\begin{aligned}
C_\alpha(u, v) = {} & u|v - u|^\alpha - \frac{u+v}{2}|v - u|^\alpha + (u + v - 1) \\
& - (u + v - 1)\big(1 - |u + v - 1|\big)^\alpha \\
& + \frac{u+v}{2}\big(1 - |u + v - 1|\big)^\alpha \\
& - \frac{\alpha}{2(\alpha+1)}\Big(\big(1 - |u + v - 1|\big)^{\alpha+1} - |v - u|^{\alpha+1}\Big) \\
= {} & W(u, v) + \frac{1}{2(\alpha+1)}\Big(\big(1 - |u + v - 1|\big)^{\alpha+1} - |v - u|^{\alpha+1}\Big).
\end{aligned}$$

Recap: Sections

Definition

Let C be a copula. Then for $u, v \in \mathbb{I}$ the functions

$$C(\cdot, v) : \mathbb{I} \to \mathbb{I}, \quad t \mapsto C(t, v),$$

$$C(u, \cdot) : \mathbb{I} \to \mathbb{I}, \quad t \mapsto C(u, t),$$

and

$$\delta_C : \mathbb{I} \to \mathbb{I}, \quad t \mapsto C(t, t)$$

are called *horizontal*, *vertical* and *diagonal section*, respectively.

Copulas with Linear Sections

Let C be a copular an suppose it has a linear horizontal section, then for any $(u, v) \in \mathbb{I}^2$ we have

$$C(u, v) = a(v)u + b(v).$$

From the boundary conditions we get

$$0 = C(0, v) = b(v) \quad \Rightarrow \quad v = C(1, v) = a(v),$$

which results in $C(u, v) = uv$. Since this holds also for the vertical section, the only copula with linear sections is the product copula.

Copulas with Quadratic Sections

Let C be a copular an suppose it has a quadratic section in u, then for any $(u, v) \in \mathbb{I}^2$ we have

$$C(u, v) = a(v)u^2 + b(v)u + c(v).$$

Again, from boundary conditions we get

$$0 = C(0, v) = c(v) \quad \Rightarrow \quad v = C(1, v) = a(v) + b(v).$$

Now, choose a function ψ such that

$$\psi(v) = -a(v) \quad \Rightarrow \quad b(v) = v - a(v) = v + \psi(v).$$

which results in

$$C(u, v) = -\psi(v)u^2 + \big(v + \psi(v)\big)u = uv + \psi(v)(1 - u)u.$$

Farlie-Gumbel-Morgenstern family

Goal: construct a symmetric copula C with quadratic sections in both u and v. As a consequence we have

$$C_\theta(u,v) = uv + \underbrace{\theta v(1-v)}_{=:\psi(v)} u(1-u),$$

where $\theta \in [-1,1]$. Then the boundary conditions for a copula are satisfied and for a rectangle $[u_1,u_2] \times [v_1,v_2] \in \mathbb{I}^2$ we have

$$V_{C_\theta} = [...] = (u_2-u_1)(v_2-v_1)\big(1+\theta(1-u_1-u_2)(1-v_1-v_2)\big),$$

which is greater or equal than 0.

How to generally choose ψ?

Theorem

Let ψ be a function with domain $\mathbb{I}$ and let C be given by

$$C(u, v) = uv + \psi(v)u(1 - u).$$

Then C is a copula if and only if

1. $\psi(0) = \psi(1) = 0$
2. ψ *satisfies the Lipschitz condition, i.e. for all* $v_1, v_2 \in \mathbb{I}$

$$|\psi(v_2) - \psi(v_1)| \leq |v_2 - v_1|.$$

Furthermore, C is absolutely continous.

Copulas with Cubic Sections

Construction is similar to copulas with quadratic sections: If C is a copula with cubic horizontal section, then

$$C(u, v) = a(v)u^3 + b(v)u^2 + c(v)u + d(v)$$

and, once again, with boundary conditions we get

$$d(v) = 0 \quad \Rightarrow \quad c(v) = v - a(v) - b(v).$$

Let $\alpha(v) = -a(v) - b(v)$ and $\beta(v) = -2a(v) - b(v)$ with $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$, then

$$C(u, v) = uv + u(1 - u)\big(\alpha(v)(1 - u) + \beta(v)u\big).$$

How to choose α and β?

Theorem

Let α, β be two functions with domain $\mathbb{I}$ satisfying $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$ and let C be given by

$$C(u, v) = uv + u(1-u)\big(\alpha(v)(1-u) + \beta(v)u\big).$$

Then C is a copula, if and only if

1. *α and β are absolutely continous.*
2. *For almost all $v \in \mathbb{I}$, either*

$$-1 \leq \alpha'(v) \leq 2 \text{ and } -2 \leq \beta'(v) \leq 1$$

or

$$\big(\alpha'(v)\big)^2 - \alpha'(v)\beta'(v) + \big(\beta'(v)\big)^2 - 3\alpha'(v) + 3\beta'(v) \leq 0.$$

Cubic Sections in both u and v?

We want to find *all* copulas that have both cubic horizontal sections and vertical sections. That is, copulas satisfying:

$$C(u,v) = uv + u(1-u)\big(\alpha(v)(1-u) + \beta(v)u\big) \tag{1}$$

and

$$C(u,v) = uv + v(1-v)\big(\gamma(u)(1-v) + \epsilon(u)v\big). \tag{2}$$

with γ,ϵ satisfying the same conditions as α, β.

Theorem

Suppose that a copula C has cubic sections in both u and v, i.e. C is given by both (1) *and* (2). *Then*

$$C(u,v) = uv + uv(1-u)(1-v)\big(A_1 v(1-u) + A_2(1-v)(1-u) + B_1 uv + B_2 u(1-v)\big),$$

where $A_1, A_2, B_1, B_2 \in \mathbb{R}$ *such that for all* $(x,y) \in \{(A_2,A_1),(B_1,B_2),(B_1,A_1),(A_2,B_2)\}$

$$-1 \leq x \leq 2 \text{ and } -2 \leq y \leq 1$$

or

$$x^2 - xy + y^2 - 3x + 3y \leq 0.$$

With this result we have an explicit expression of α, β, γ and ϵ:

$$
\begin{aligned}
\alpha(v) &= v(1-v)\big(A_1 v + A_2(1-v)\big)\\
\beta(v) &= v(1-v)\big(B_1 v + B_2(1-v)\big)\\
\gamma(u) &= u(1-u)\big(B_2 u + A_2(1-u)\big)\\
\epsilon(u) &= u(1-u)\big(B_1 u + A_1(1-u)\big)
\end{aligned}
$$

Example

Let a, b be constants such that $A_1 = B_2 = a - b$ and $A_2 = B_1 = a + b$ satisfy the previos theorem's conditions. Then, we have

$$\alpha(v) = v(1 - v)(a + b - 2bv),$$
$$\beta(v) = v(1 - v)(a - b + 2bv)$$

and $\gamma \equiv \alpha$, $\epsilon \equiv \beta$. The resulting copula is given by

$$C_{a,b}(u, v) = uv + uv(1 - u)(1 - v)\big(a + b(1 - 2u)(1 - 2v)\big).$$

Recap: Dual of a Copula

Definition

Let C be a copula and $\delta_C : \mathbb{I} \to \mathbb{I}$, $\delta_C(t) := C(t, t)$ its diagonal section. Then the *dual* of C is given by the function

$$\tilde{\delta}_C : \mathbb{I} \to \mathbb{I} \;\; \tilde{\delta}_C(t) = 2t - \delta_C(t).$$

Distribution Functions of Order Statistics

Let X, Y be two random variables with common distribution function F and copula C. Then following from Sklar's theorem

$$\mathbb{P}\big(\max(X,Y) \leq t\big) = \mathbb{P}(X \leq t, Y \leq t) = C\big(F(t), F(t)\big) = \delta_C\big(F(t)\big)$$

and

$$\begin{aligned}\mathbb{P}\big(\min(X,Y) \leq t\big) &= \mathbb{P}(X \leq t) + \mathbb{P}(Y \leq t) - \mathbb{P}(X \leq t, Y \leq t)\\ &= 2F(t) - \delta_C\big(F(t)\big) = \tilde{\delta}_C\big(F(t)\big)\end{aligned}$$

In the following we call any function $\delta : \mathbb{I} \to \mathbb{I}$ that satisfies

(1) $\delta(1) = 1$

(2) $0 \leq \delta(t_2) - \delta(t_1) \leq 2(t_2 - t_1)$, for any $t_1, t_2 \in \mathbb{I}$, $t_1 \leq t_2$

(3) $\delta(t) \leq t$ for any $t \in \mathbb{I}$

a *diagonal*.

Constructing Copulas with diagonals

Theorem

Let δ be any diagonal and set

$$C(u,v) := \min\left(u, v, \frac{1}{2}(\delta(u) + \delta(v))\right).$$

Then C is a copula whose diagonal section is δ.

These copulas are called diagonal copulas. Note, that for $\delta(t) = t$

$$C(u,v) := \min\left(u, v, \frac{u+v}{2}\right) = \min(u,v) = M(u,v).$$

Joint Distribution Functions of Order Statistics

Theorem

Suppose X and Y are continous random variables with copula C and a common marginal distribution. Then the joint distribution function H of $\min(X, Y)$ and $\max(X, Y)$ is the Fréchet-Hoeffding upper bound M, i.e.

$$H(x, y) = M\big((\tilde{\delta}_C(x), \delta_C(y)\big),$$

if and only if C is a diagonal copula.