Archimedean Copulas

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Motivation

Let X and Y be continuous random variables, with joint distribution function H and marginal distribution functions F and G.
Saw cases in which

$$\lambda(H(x,y))) = \lambda(F(x))\lambda((G(y))$$

for a function λ.

Chapter 1: Definitions

Motivation

Let X and Y be continuous random variables, with joint distribution function H and marginal distribution functions F and G.
Saw cases in which

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for a function λ.
For copulas, and if we define $\varphi(t) = -\log(\lambda(t))$ we have

$$\begin{aligned} \varphi(C(u,v)) &= \varphi(u) + \varphi(v) \\ \Rightarrow \qquad C(u,v) &= \varphi^{[-1]}(\varphi(u) + \varphi(v)) \end{aligned}$$

Definition 1.1.

Let $\varphi : \mathbf{I} \rightarrow [0, \infty]$ continuous, strictly decreasing such that $\varphi(1) = 0$. The pseudo-inverse $\varphi^{[-1]} : [0, \infty] \rightarrow \mathbf{I}$ is defined as

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t \leq \infty \end{cases}$$

Lemma 1.2.(Properties of pseudo-inverse)
Let $\varphi^{[-1]}$ be defined as above. Then

(i) $\varphi^{[-1]}$ is continuous, non-increasing on $[0, \infty]$ and strictly decreasing on $[0, \varphi(0)]$.

(ii) $\forall t \in \mathbf{I} : \varphi^{[-1]}(\varphi(t)) = t$

(iii) $\forall t \in \mathbf{I} : \varphi(\varphi^{[-1]}(t)) = \min(t, \varphi(0))$

(iv) $\varphi(0) = \infty \Rightarrow \varphi^{[-1]} = \varphi^{-1}$

Lemma 1.3.

Let $\varphi : \mathbf{I} \to [0, \infty]$ continuous, strictly decreasing such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$$

Then C satisfies the boundary conditions for a copula that is for every $u, v \in \mathbf{I}$

$$C(u, 0) = C(0, v) = 0$$

and

$$C(u, 1) = u, \quad C(1, v) = v.$$

Proof.
Let $u \in \mathbf{I}$. By definition of $\varphi^{[-1]}$ it follows that

$$C(u,0) = \varphi^{[-1]}(\varphi(u) + \varphi(0)) = 0$$

and

$$C(u,1) = \varphi^{[-1]}(\varphi(u) + \varphi(1)) = \varphi^{[-1]}(\varphi(u)) = u$$

By symmetry the claim follows.

Lemma 1.4.

Let $\varphi : \mathbf{I} \to [0, \infty]$ continuous, strictly decreasing such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$$

Then C is 2-increasing if and only if for all $v \in \mathbf{I}$:

$$u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1$$

Proof.

"⇒" Note that $C(u_2, v) - C(u_1, v) \leq u_2 - u_1$ is equivalent to $V_C([u_1, u_2] \times [v, 1]) \geq 0$, which is true if C is 2-increasing.

"⇐" Now assume that for all $v \in \mathbf{I}$:

$$u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1.$$

Let $v_1, v_2 \in \mathbf{I}$ with $v_1 \leq v_2$. By Lemma 1.2.

$$C(0, v_2) = 0 \leq v_1 \leq v_2 = C(1, v_2).$$

Therefore by continuity of C there exists a t in $\mathbf{I}$ with $C(t, v_2) = v_1$.

$$\begin{aligned} C(u_2, v_1) - C(u_1, v_1) = & \quad \varphi^{[-1]}(\varphi(u_2) + \varphi(v_2) + \varphi(t)) - \\ & \quad \varphi^{[-1]}(\varphi(u_1) + \varphi(v_2) + \varphi(t)) \\ = & \quad C(C(u_2, v_2), t) - C(C(u_1, v_2), t) \\ \leq & \quad C(u_2, v_2) - C(u_1, v_2) \end{aligned}$$

The claim follows.

Theorem 1.5.

Let $\varphi : \mathbf{I} \to [0, \infty]$ continuous, strictly decreasing such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse. Let $C : \mathbf{I}^2 \to \mathbf{I}$ defined as before. Then C is a copula if and only if φ is convex.

In this case C is called Archimedean copula, and φ is called generator of C. If $\varphi(0) = \infty$, φ is a strict generator and C is called strict Archimedean copula.

Proof.

By the preceding Lemmata it is enough to show that

$$\forall v \in \mathbf{I} : (u_1 \leq u_2 \Rightarrow C(u_2, v) - C(u_1, v) \leq u_2 - u_1) \Leftrightarrow \varphi \text{ convex}$$

This is a bit technical so we skip the proof.

Example 1.6.

(a) Let $\varphi(t) = -\log(t)$, $t \in [0, 1]$. It follows that $C(u, v) = uv$.

(b) Let $\varphi(t) = 1 - t$, $t \in [0, 1]$. It follows that
$C(u, v) = \max(u + v - 1, 0)$.

Chapter 2: One-parameter families

- By Theorem 1.5. Archimedean copulas can be contructed by finding a suitable generator
- Archimedean Copulas easy to construct
- Wide variety of dependence structures
- Two example one-parameter families:

$C_\theta(u,v)$	$\varphi_\theta(t)$	$\theta \in$	Special cases
$C_\theta(u,v) = \exp(-[(-\log u)^\theta] + (-\log(v)^\theta)]^{1/\theta})$	$\varphi_\theta(t) = (-\log t)^\theta$	$[1,\infty)$	$C_1 = \prod,\ C_\infty = M$
$C_\theta(u,v) = \max(\theta uv + (1-\theta)(u+v-1), 0)$	$\varphi_\theta(t) = -\log(\theta t + (1-\theta))$	$(0,1]$	$C_0 = W,\ C_1 = \prod$

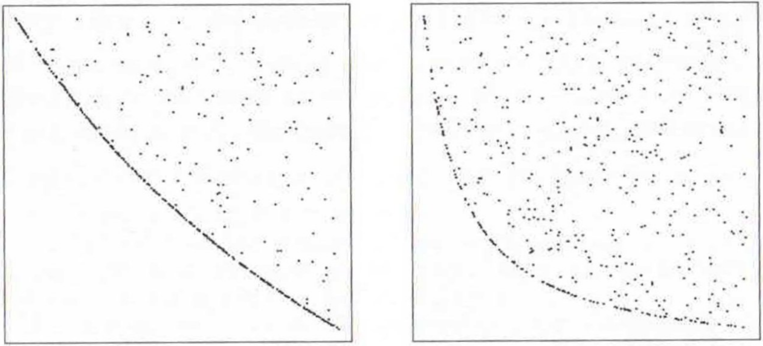


Figure: Scatterplots, $\theta = 0.4$ (left) and $\theta = 0.9$ (right)

Chapter 3: Fundamental Properties

For simplicity let Ω denote the set of continuous strictly decreasing convex functions $\varphi : \mathbf{I} \to [0, \infty]$ with $\varphi(1) = 0$.

Theorem 3.1.

Let C be an Archimedean copula generated by $\varphi \in \Omega$. Let $K_C(t)$ denote the C-measure of the set

$$\{(u, v) \in \mathbf{I}^2 | C(u, v) \leq t\} = \{(u, v) \in \mathbf{I}^2 | \varphi(u) + \varphi(v) \geq \varphi(t)\}.$$

Then for any t in $\mathbf{I}$

$$K_C(t) = t - \frac{\varphi(t)}{\varphi'(t+)}$$

Proof.

Let t be in $(0,1)$, and set $w = \varphi(t)$. Let $n \in \mathbb{N}$. Let $W := \{0, \frac{w}{n}, \dots, \frac{wn}{n}\}$ be a partition of $[0, w]$ and $T := \{t = t_0, \dots, t_n = 1\}$ be a partition of $[t, 1]$ with

$$t_{n-k} = \varphi^{[-1]}(\frac{kw}{n}), \quad k = 0, 1, \dots, n$$

It follows that

$$C(t_j, t_k) = \varphi^{[-1]}(\varphi(t_j) + \varphi(t_k)) = \varphi^{[-1]}(w + \frac{n-j-k}{n} w)$$

Especially $C(t_j, t_{n-j}) = t$.

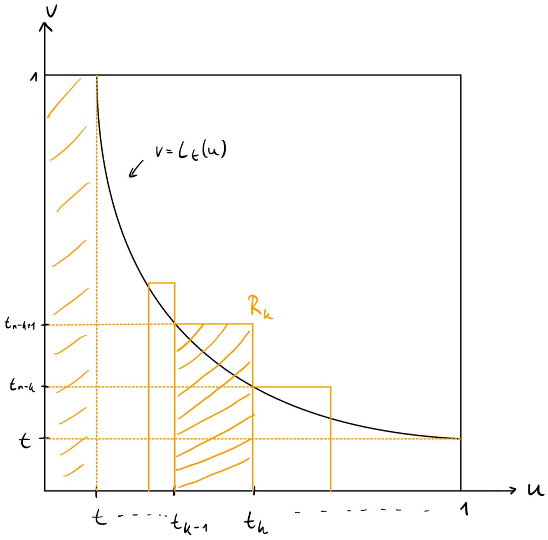
Proof.

Denote $[t_{k-1}, t_k] \times [0, t_{n-k+1}]$ by R_k, and let $S_n = \cup_{k=1}^{n} R_k$.

Then we have $K_C(t)$ is the sum of the C-measure of $[0, t] \times \mathbf{I}$ and $\lim_{n\to\infty} V_C(S_n)$, since

$$0 \leq t_1 - t_0 \leq \ldots \leq t_n - t_{n-1}$$

and $\lim_{n\to\infty} t_n - t_{n-1} = 0$.


v
1
$v = L_t(u)$
R_k
t_{n-k+1}
t_{n-k}
t
u
t
t_{k-1}
t_k
1

Proof.

And for each k

$$V_C(R_k) = C(t_k, t_{n-k+1}) - t = \varphi^{[-1]}(w - \frac{w}{n}) - \varphi^{[-1]}(w)$$

and hence

$$V_C(S_n) = \sum_{k=1}^{n} V_C(R_k) = -w\left[\frac{\varphi^{[-1]}(w) - \varphi^{[-1]}(w - w/n)}{w/n}\right]$$

The claim follows by taking the limit $n \to \infty$.

Corollary 3.2.

Let C be an Archimedean copula generated by $\varphi \in \Omega$. Let $K_C'(s,t)$ denote the C-measure of the set

$$\{(u,v) \in \mathbf{I}^2 | u \leq s, C(u,v) \leq t\}$$

Then for any $(s,t) \in \mathbf{I}^2$

$$K_C'(s,t) = \begin{cases} s, & s \leq t \\ t - \frac{\varphi(t)-\varphi(s)}{\varphi'(t^+)} & s > t \end{cases}$$

Corollary 3.3.

Let U and V be uniform (0,1) random variables with joint distribution function C generated by $\varphi \in \Omega$. Then the function K_C is the distribution function of $C(U, V)$. Furthermore, the function K'_C is the joint distribution function of U and $C(U, V)$.

Chapter 4: Order and Limiting Cases

Definition 4.1.

Let C_1 and C_2 be Copulas, we say C_1 is smaller than C_2 (or C_2 is larger than C_1), and write $C_1 \prec C_2$ (or $C_2 \succ C_1$) if $C_1(u, v) \leq C_2(u, v)$ for all u, v in $\mathbf{I}$.

We say a family $\{C_\theta\}$ of copulas is positively ordered if

$$\alpha \leq \beta \Rightarrow C_\alpha \prec C_\beta.$$

The family is negatively ordered if

$$\alpha \leq \beta \Rightarrow C_\alpha \succ C_\beta$$

Example 4.2. Consider this one-parameter family of Archimedean copulas:

$C_\theta(u,v)$	$\varphi_\theta(t)$	$\theta \in$
$C_\theta(u,v) = \theta / \log(e^{\theta/u} + e^{\theta/v} - e^{\theta})$	$\varphi_\theta(t) = e^{\theta/t} - e^{\theta}$	$(0,\infty)$

Now let $\theta_1, \theta_2 \in (0,\infty)$, $\theta_1 \leq \theta_2$. Is there a relation between

$$\frac{\theta_1}{\log(e^{\theta_1/u} + e^{\theta_1/v} - e_1^{\theta})} \text{ and } \frac{\theta_2}{\log(e^{\theta_2/u} + e^{\theta_2/v} - e_2^{\theta})}?$$

Definition 4.3.

A function $f : [0, \infty) \to \mathbb{R}$ is subadditive if for all $x, y \in [0, \infty)$

$$f(x + y) \leq f(x) + f(y).$$

Theorem 4.4.

Let C_1 and C_2 be Archimedean copulas generated by φ_1 and φ_2 in Ω. Then $C_1 \prec C_2$ if and only if $\varphi_1 \circ \varphi_2^{[-1]}$ is subadditive.

Proof. Let $f = \varphi_1 \circ \varphi_2^{[-1]}$. f is continuous, nondecreasing, and $f(0) = 0$. Per definitionem, $C_1 \prec C_2$ if and only if for all u, v in $\mathbf{I}$,

$$\varphi_1^{[-1]}(\varphi_1(u) + \varphi_1(v)) \leq \varphi_2^{[-1]}(\varphi_2(u) + \varphi_2(v)).$$

Let $x = \varphi_2(u)$ and $y = \varphi_2(v)$, then the above is equivalent to

$$\varphi_1^{[-1]}(f(x) + f(y)) \leq \varphi_2^{[-1]}(x + y) \tag{1}$$

for all x, y in $[0, \varphi_2(0)]$.In addition if $x > \varphi_2(0)$ or $y > \varphi_2(0)$, then (1) reduces to $0 \leq 0$.

Proof.

$$\varphi_1^{[-1]}(f(x)+f(y)) \leq \varphi_2^{[-1]}(x+y) \tag{1}$$

"⇒" Now let $C_1 \prec C_2$. The claim follows by applying φ_1 to both sides of (1).
"⇐" Conversely let f be subadditive we can apply $\varphi_1^{[-1]}$ to

$$f(x+y) \leq f(x)+f(y)$$

which yields the desired relation.

Theorem 4.5.

Let $\{C_\theta | \theta \in \Theta\}$ be a family of Archimedean copulas with differentiable generators φ_θ in Ω. Then $C = \lim C_\theta$ is an Archimedean copula if and only if there exists a function φ in Ω such that for all s, t in $(0, 1)$:

$$\lim \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \frac{\varphi(s)}{\varphi'(t)}$$

where lim denotes the appropriate one-sided limit as θ approaches an end point of the parameter interval. The generator of C is φ.

Proof.

Let (U_θ, V_θ) be uniform (0,1) random variables with joint distribution function C_θ, let K_θ denote the distribution function of $C_\theta(U_\theta, V_\theta)$ and let K'_θ denote the joint distribution function of U_θ and $C_\theta(U_\theta, V_\theta)$. By Corollaries 3.2. and 3.3. we get

$$K'_\theta(s,t) = t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)} + \frac{\varphi_\theta(s)}{\varphi'_\theta(t)}$$

for $0 < t < s < 1$ and

$$K_\theta(t) = t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}$$

for all t in $\mathbf{I}$

Proof.

Now let (U, V) be uniform (0,1) random variables with joint distribution function C, let K be the distribution function of $C(U, V)$ and let K' denote the joint distribution function of U and $C(U, V)$.

Assume $C = \lim C_\theta$ is Archimedean with generator φ. So

$$\lim t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)} = \lim K_\theta(t) = K(t) = t - \frac{\varphi(t)}{\varphi'(t)} \tag{1}$$

for $t \in \mathbf{I}$. This proves the claim for $s = t$.

For $0 < t < s < 1$. It now holds that

$$\lim t - \frac{\varphi_\theta(t)}{\varphi_\theta'(t)} + \frac{\varphi_\theta(s)}{\varphi_\theta'(t)} = \lim K_\theta'(s,t) = K'(s,t) = t - \frac{\varphi(t)}{\varphi'(t)} + \frac{\varphi(s)}{\varphi\prime(t)}$$

And with (1) we get

$$\lim \frac{\varphi_\theta(s)}{\varphi_\theta'(t)} = \frac{\varphi(s)}{\varphi'(t)}.$$

Conversely, assume that for all s, t in $(0,1)$:

$$\lim \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \frac{\varphi(s)}{\varphi'(t)}$$

We therefore have positive constants c_θ such that for all $t \in (0,1]$, $\lim c_\theta \varphi_\theta(t) = \varphi(t)$. So

$$\lim \varphi_\theta^{[-1]}(\frac{\cdot}{c_\theta}) = \varphi^{[-1]}(\cdot).$$

It follows

$$\lim \varphi_\theta^{[-1]}[\varphi_\theta(u) + \varphi_\theta(v)] = \varphi^{[-1]}[\varphi(u) + \varphi(v)]$$

for fixed $u, v \in \mathbf{I}$.

Example 4.6.

- Let $\varphi_\theta(t) = -\log(\theta t + (1-\theta))$, $\theta \in (0,1]$. Using L'Hospital:

$$\lim_{\theta \to 0+} \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \lim_{\theta \to 0+} \frac{\log(\theta s + (1-\theta))}{\theta/(\theta t + (1-\theta))} = s - 1$$

So it follows $C_0 = W$.

Example 4.6.

- Let $\varphi_\theta(t) = -\log(\theta t + (1-\theta))$, $\theta \in (0,1]$. Then we have

$$\lim_{\theta \to 1-} \frac{\varphi_\theta(s)}{\varphi'_\theta(t)} = \lim_{\theta \to 1-} \frac{\log(\theta s + (1-\theta))}{\theta/(\theta t + (1-\theta))} = t \log s$$

So it follows $C_1 = \prod$.

Chapter 5: Two-parameter families

Aim: Construct two-parameter families of Archimedean copulas

One approach is to compose generators with the power function $t \mapsto t^{\theta}$, $\theta > 0$.

Theorem 5.1. Let $\varphi \in \Omega$, let $\alpha, \beta > 0$ and define

$$\varphi_{\alpha,1}(t) = \varphi(t^{\alpha}) \qquad \varphi_{1,\beta}(t) = [\varphi(t)]^{\beta}$$

If $\beta \geq 1$, then $\varphi_{1,\beta} \in \Omega$. If α is in $(0, 1]$, then $\varphi_{\alpha,1} \in \Omega$.
If φ is twice differentiable and $t\varphi'(t)$ is nondecreasing on $(0, 1)$, then $\varphi_{\alpha,1}$ is an element of Ω for all $\alpha > 0$.

To construct a two-parameter family of copulas we can now define

$$\varphi_{\alpha,\beta}(t) = [\varphi(t^\alpha)]^\beta$$

and note that $\varphi_{\alpha,\beta} \in \Omega$ if we choose α and β as in the Theorem above.

Example 5.2. Let $\varphi(t) = 1 - t$ and using our approach we define $\varphi_{\alpha,\beta}(t) = (1 - t^\alpha)^\beta$ for $\alpha \in (0, 1]$, $\beta \geq 1$. This generates

$$C_{\alpha,\beta}(u, v) = \max\left(\left[1 - ((1 - u^\alpha)^\beta + (1 - v^\alpha)^\beta)^{1/\beta}\right]^{1/\alpha}, 0\right)$$