Concordance and Dependence Properties

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Concordance

Motivation example: Let X be a symmetric random variable. Then X and $Y = X^2$ are obviously dependent. What about correlation?

$$\mathbf{Corr}(X, Y) = \mathbf{E}(XX^2) - \mathbf{E}X^2\mathbf{E}X = 0 - 0 = 0.$$

- Informally, a pair of random variables are concordant if “large” values of one tend to be associated with “large” values of the other and “small” values of one with “small” values of the other.
- Formally, let (x_i, y_i) and (x_j, y_j) denote two observations from a vector (X, Y). We say that (x_i, y_i) and (x_j, y_j) are concordant if $x_i < x_j$ and $y_i < y_j$, or if $x_i > x_j$ and $y_i > y_j$. We say that (x_i, y_i) and (x_j, y_j) are discordant if $x_i < x_j$ and $y_i > y_j$ or if $x_i > x_j$ and $y_i < y_j$.
- The alternate formulation: (x_i, y_i) and (x_j, y_j) are concordant if $(x_i - x_j)(y_i - y_j) > 0$ and discordant if $(x_i - x_j)(y_i - y_j) < 0$.

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Kendall's τ

- Let $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ denote a random sample of n observations from a vector (X, Y) of continuous random variables.
- Each distinct pair is either concordant or discordant. Let c denote the number of concordant pairs and d the number of discordant pairs. Then Kendall's τ for the sample is defined as
$$t = \frac{c-d}{c+d} = \frac{c-d}{\binom{n}{2}}.$$
- Let (X_1, Y_1) and (X_2, Y_2) be i.i.d. random vectors, each with joint distribution function H. Then the population version of Kendall's τ is defined as the probability of concordance minus the probability of discordance:
$$\tau = \tau_{X,Y} = \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

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Theorem 1

Let (X_1, Y_1) and (X_2, Y_2) be independent vectors of continuous random variables with joint distribution functions H_1 and H_2, respectively, with common margins F (of X_1 and X_2) and G (of Y_1 and Y_2). Let C_1 and C_2 denote the copulas of (X_1, Y_1) and (X_2, Y_2), respectively, so that $H_1(x, y) = C_1(F(x), G(y))$ and $H_2(x, y) = C_2(F(x), G(y))$. Let Q denote the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2), i.e., let

$$Q = \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Then

$$Q = Q(C_1, C_2) = 4 \int_{I^2} C_2(u, v) dC_1(u, v) - 1.$$

Proof: The random variables are continuous:
$\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) < 0] = 1 - \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0]$ and hence
$Q = 2\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - 1.$

$$\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] = \mathbf{P}[X_1 > X_2, Y_1 > Y_2] + \mathbf{P}[X_1 < X_2, Y_1 < Y_2].$$

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Corollary 2

Let C_1, C_2, and Q be as given in Theorem 1. Then

1. *Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$.*
2. *Q is non-decreasing in each argument: if $C_1 < C_1'$ and $C_2 < C_2'$ for all $(u, v) \in I^2$, then $Q(C_1, C_2) \leq Q(C_1', C_2')$.*
3. *Copulas can be replaced by survival copulas in Q, i.e., $Q(C_1, C_2) = Q(\hat{C}_1, \hat{C}_2)$.*

Example 3

$M(u, v) = \min(u, v)$, $W(u, v) = \max(u + v - 1, 0)$, $\Pi(u, v) = uv$.

$$\begin{array}{lll} Q(W,W) = -1, & Q(W,\Pi) = -1/3, & Q(W,M) = 0, \\ Q(\Pi,W) = -1/3, & Q(\Pi,\Pi) = 0, & Q(\Pi,M) = 1/3, \\ Q(M,W) = 0, & Q(M,\Pi) = 1/3, & Q(M,M) = 1. \end{array}$$

Let C be an arbitrary copula, then
$Q(W, C) \in [-1, 0]$, $\quad Q(\Pi, C) \in [-1/3, -1/3]$, $\quad Q(M, C) \in [0, 1]$.

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Let C_1, C_2, and Q be as given in Theorem 1. Then

1. *Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$.*
2. *Q is non-decreasing in each argument: if $C_1 < C_1'$ and $C_2 < C_2'$ for all $(u, v) \in I^2$, then $Q(C_1, C_2) \leq Q(C_1', C_2')$.*
3. *Copulas can be replaced by survival copulas in Q, i.e., $Q(C_1, C_2) = Q(\hat{C}_1, \hat{C}_2)$.*

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$M(u, v) = \min(u, v)$, $W(u, v) = \max(u + v - 1, 0)$, $\Pi(u, v) = uv$.

$$\begin{array}{lll} Q(W, W) = -1, & Q(W, \Pi) = -1/3, & Q(W, M) = 0, \\ Q(\Pi, W) = -1/3, & Q(\Pi, \Pi) = 0, & Q(\Pi, M) = 1/3, \\ Q(M, W) = 0, & Q(M, \Pi) = 1/3, & Q(M, M) = 1. \end{array}$$

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Kendall's τ for a copula

Theorem 4

Let X and Y be continuous random variables whose copula is C. Then the population version of Kendall's τ for X and Y is given by

$$\tau_{X,Y} = \tau_C = Q(C, C) = 4 \int_{I^2} C(u, v) dC(u, v) - 1.$$

Note that $\tau_C = 4\mathbf{E}(C(U, V)) - 1$, where $U, V \sim Unif[0, 1]$ with $(U, V) \sim C$.

Theorem 5 (Li et al. 2002)

Let C_1 and C_2 be copulas. Then

$$\int_{I^2} C_1(u, v) dC_2(u, v) = \frac{1}{2} - \int_{I^2} \frac{\partial C_1(u, v)}{\partial u} \frac{\partial C_2(u, v)}{\partial v} dudv.$$

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Examples

- The Farlie-Gumbel-Morgernstern family: $C_\theta(u,v) = uv + \theta uv(1-u)(1-v)$, where θ is in [–1, 1]. We have

$$dC_\theta(u,v) = (1 + \theta(1-2u)(1-2v))dudv.$$

Then

$$\int_{I^2} C_\theta(u,v)dC_\theta(u,v) = \frac{1}{4} + \frac{\theta}{18}$$

and $\tau_C = 2\theta/9 \in [-2/9, 2/9]$.

- The Fréchet family: $C_{\alpha,\beta} = \alpha M + (1-\alpha-\beta)\Pi + \beta W$, where $\alpha, \beta \geq 0, \alpha + \beta \leq 1$. Then

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Archimedean copulas

$C(u,v) = \varphi^{(-1)}(\varphi(u) + \varphi(v))$, where φ is a continuous, strictly decreasing, convex function.

Corollary 6 (Genest and MacKay 1986)

Let X and Y be random variables with an Archimedean copula C generated by φ in Ω. The population version τ_C of Kendall's τ for X and Y is given by

$$\tau_C = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt.$$

Proof: Let U and V be uniform(0,1) distributed random variables with joint distribution function C, and let K_C denote the distribution function of $C(U,V)$. Then

$$\tau_C = 4\mathbf{E}(C(U,V)) - 1 = 4\int_0^1 t dK_C(t) - 1 = 3 - 4\int_0^1 K_C(t)dt.$$

$K_C(t)$ is the $C-$measure of the set $\{(u,v) \in I^2 : C(u,v) \leq t\}$, or, equivalently, of the set $\{(u,v) \in I^2 : \varphi(u) + \varphi(v) \geq \varphi(t)\}$,
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Spearman's ρ

Theorem 7

Let X and Y be continuous random variables whose copula is C. Then the population version of Spearman's ρ for X and Y is given by

$$\rho_{X,Y} = \rho_C = 3Q(C,\Pi) = 12\int_{I^2} uv\,dC(u,v) - 3 = 12\int_{I^2} C(u,v)\,du\,dv - 3.$$

Spearman's rho is often called the "grade" correlation coefficient. Grades are the population analogs of ranks, that is, if x and y are observations from two random variables X and Y with distribution functions F and G, respectively, then the grades of x and y are given by $u = F(x)$ and $v = G(y)$.
Note that the grades (u and v) are observations from the uniform (0,1) random variables $U = F(X)$ and $V = G(Y)$ whose joint distribution function is C.
The coefficient "3" is a "normalization" constant, because $Q(C,\Pi) \in [-1/3, 1/3]$.

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Analogue with correlation

Let $U = F(X)$ and $V = G(Y)$, then

$$\rho_{X,Y} = \rho_C = 12\int_{I^2} uv dC(u,v) - 3 = 12\mathbf{E}(UV) - 3$$
$$= \frac{\mathbf{E}(UV) - 1/4}{1/12} = \frac{\mathbf{E}(UV) - \mathbf{E}(U)\mathbf{E}(V))}{\sqrt{Var(U)}\sqrt{Var(V)}} = \mathbf{Corr}(U,V).$$

- Spearman's ρ for a pair of continuous random variables X and Y is identical to Pearson's product-moment correlation coefficient for the grades of X and Y.
- From $\rho_C = 12\int_{I^2}[C(u,v) - uv]dudv$ we have that ρ_C is proportional to the signed volume between the graphs of the copula C and the product copula Π.
- Thus ρ_C is a measure of "average distance" between the distribution of X and Y (as represented by C) and independence (as represented by the copula Π).

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Examples

- Farlie-Gumbel-Morgernstern family: $C_\theta(u,v) = uv + \theta uv(1-u)(1-v)$, where θ is in [–1, 1]. Then $\tau_C = 2\theta/9 \in [-2/9, 2/9]$. Compute

$$\rho_C = 12 \int_{I^2} [uv + \theta uv(1-u)(1-v)]\, dudv - 3$$
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Let $X \sim N(0,1)$ and $Y = X^2$.

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Hoeffding's lemma 1940

Let X and Y be random variables with joint distribution function H and margins F and G, such that $\mathbf{E}(|X|)$, $\mathbf{E}(Y)$, and $\mathbf{E}(|XY|)$ are all finite. Then

$$\mathbf{Cov}(X,Y) = \int_{\mathbb{R}^2} [H(x,y) - F(x)G(y)]dxdy.$$

Moreover,

$$\begin{aligned}\mathbf{Cov}(X,Y) &= \int_{\mathbb{R}^2} [C(F(x),G(y)) - F(x)G(y)]dxdy \\ &= \int_{I^2} [C(u,v) - uv]dF^{-1}(u)dG^{-1}(v).\end{aligned}$$

Measures of dependence

- Schweizer and Wolff's σ is given by

$$\sigma_{X,Y} = \int_{I^2} |C(u,v) - uv| dudv$$

- For any p, $1 \geq p < \infty$, the L_p distance between C and Π is given by

$$\left(k_p \int_{I^2} |C(u,v) - uv|^p \, dudv \right)^{1/p},$$

where k_p is a constant chosen so that the quantity is 1 when $C = M$ or W