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Geometric Methods of Constructing Copulas

Seminar Paper

Copulas and Their Applications

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1 Preliminaries

Since this paper heavily relies on the following terms, let us recall the following definitons.

Definition 1.1 (Copulas (see [4]))
A two-dimensional copula is a function $C : \mathbb{I}^2 \to \mathbb{I}$, where $\mathbb{I} = [0, 1]$, with the following properties:

1. *For every $(u, v) \in \mathbb{I}^2$, it holds that*

$$C(u, 0) = C(0, v) = 0$$

and

$$C(u, 1) = u \text{ and } C(1, v) = v.$$

2. *C is 2-increasing, i.e. for every $u_1, u_2, v_1, v_2 \in \mathbb{I}$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, it holds that*

$$V_C\Big([u_1, u_2] \times [v_1, v_2]\Big) = C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_2). \tag{1.1}$$

We refer to V_C in (1.1) as the C-volume of the rectangle $[u_1, u_2] \times [v_1, v_2]$.

Definition 1.2 (Support of a Copula (see [4]))
Let C be a copula. Then its support $S(C)$ is defined by

$$S(C) := \{A \subset \mathbb{I}^2 : A \text{ open and } V_C(A) = 0\}^c.$$

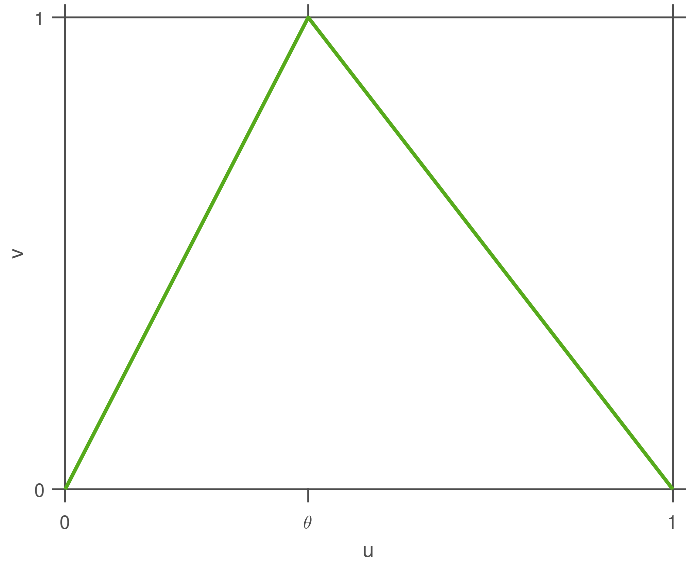
A copula C is said to be singular if it has support $S(C)$ with Lebesgue measure 0, i.e.

$$\lambda\Big(S(C)\Big) = 0.$$

2 Copulas with Prescribed Support

In this chapter we will construct singular copulas with a given support by only using the definition of a copula.

Example 2.1 (see [4])
Let $\theta \in [0,1]$ and suppose that the support of the desired copula C_θ is given by the lines connecting the points $(0,0),(\theta,1)$ and $(\theta,1),(1,0)$, i.e. the graph of $S(C_\theta)$ looks like this:



Note, that if $\theta = 1$ or $\theta = 0$, we have $C_1 = M$ and $C_0 = W$, with M and W denoting the upper and lower Fréchet-Hoeffding Bounds, respectively. We now use the fact that in order for C_θ to be a copula, it has to hold that $V_{C_\theta}(B) \geq 0$ for all rectangles $B \subset \mathbb{I}^2$ and that a rectangle not intersecting the graph of the support has a C_θ-volume of 0. For $(u,v) \in \mathbb{I}^2$, there are three cases to consider:

1. $u \leq \theta v$,
   i.e. (u, v) *lies above the left segment of the graph. Then,*

$$\begin{aligned} C_\theta(u,v) &= V_{C_\theta}([0,u] \times [0,v]) \\ &= V_{C_\theta}([0,u] \times [0,v]) + \underbrace{V_{C_\theta}([0,u] \times [v,1])}_{=0} \\ &= V_{C_\theta}([0,u] \times [0,1]) \\ &= C_\theta(u,1) \\ &= u \end{aligned}$$

2. $u > \theta v$ *and* $u < 1 - (1-v)\theta$,
   i.e (u, v) *lies below both segments of the graph. Then, by using the same tricks as before, we get*

$$\begin{aligned} C_\theta(u,v) &= V_{C_\theta}([0,u] \times [0,v]) \\ &= V_{C_\theta}([0,\theta v] \times [0,v]) + \underbrace{V_{C_\theta}([\theta v,u] \times [0,v])}_{=0} \\ &= C_\theta(\theta v, v) \\ &= V_{C_\theta}([0,\theta v] \times [0,v]) \\ &= V_{C_\theta}([0,\theta v] \times [0,v]) + \underbrace{V_{C_\theta}([0,\theta v] \times [v,1])}_{=0} \\ &= V_{C_\theta}([0,\theta v] \times [0,1]) \\ &= C_\theta(\theta v, 1) \\ &= \theta v \end{aligned}$$

3. $u \geq 1 - (1-v)\theta$,
   i.e. (u, v) *lies above the second segment of the graph. Here, it holds that*

$$V_{C_\theta}([u,1] \times [v,1]) = 0$$

   and

$$V_{C_\theta}([u,1] \times [v,1]) = C_\theta(u,v) - u - v + 1,$$

   which results in

$$C_\theta(u,v) = u + v - 1.$$

The resulting copula C_θ *is given by:*

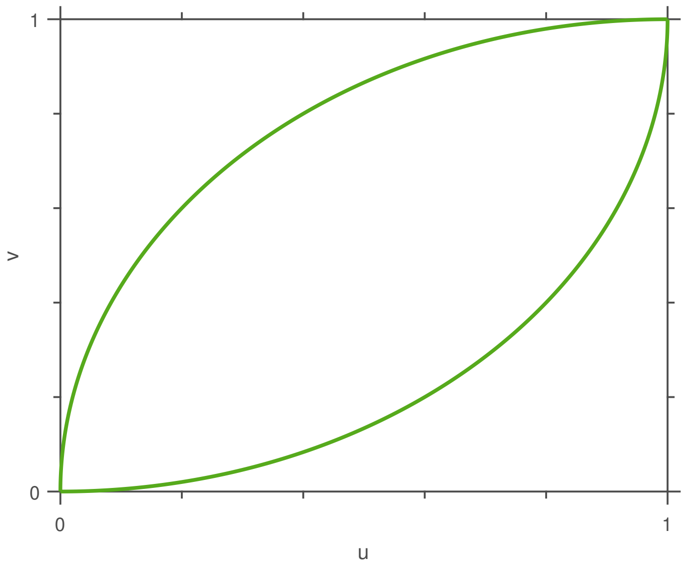
$$C_\theta(u,v) = \begin{cases} u & , \textit{ if } 0 \leq u \leq \theta v \leq \theta \\ \theta v & , \textit{ if } 0 \leq \theta v < u < 1 - (1-\theta)v \\ u + v - 1, & \textit{if } \ \theta \leq 1 - (1-\theta)v \leq u \leq 1 \end{cases}$$

Next, we want to construct a symmtric copula.

Example 2.2 (see [4])
Let the prescribed support be given by the set

$$S(C) := \{(u,v) \in \mathbb{I}^2 : u^2 + v^2 = 2u\} \cup \{(u,v) \in \mathbb{I}^2 : u^2 + v^2 = 2v\},$$

whose graph looks like this:



Again, we have to consider different cases. First, let $u^2+v^2 > 2\min(u,v)$*, i.e.* (u,v) *lies above or below the upper and lower quarter circle, respectively. Since for* $u^2+v^2 > 2u$ *it must hold that*

$$V_C([0,u] \times [v,1]) = u - C(u,v) \overset{!}{=} 0,$$

we have

$$C(u,v) = u \tag{2.1}$$

and analogue for $u^2 + v^2 > 2v$*:*

$$C(u,v) = v.$$

With the Fréchet-Hoeffding upper boundary we get

$$C(u,v) \le M(u,v) := \min(u,v) \quad \Rightarrow \quad C(u,v) = M(u,v).$$

Next let (u,v) *lie below the upper quarter circle but above the diagonal. That is,* $u \le v$ *and* $u^2 + v^2 \le 2u$*. Here, it must hold that*

$$V_C([u,v] \times [u,v]) = 0 \quad \Leftrightarrow \quad C(u,v) + C(v,u) = C(u,u) + C(v,v)$$

and since we have symmetry

$$2C(u,v) = C(u,u) + C(v,v). \tag{2.2}$$

The same holds for $v \leq u$ and $u^2 + v^2 \leq 2v$.
Now, assume $u^2 + v^2 = 2u$, i.e. (u, v) lies on the upper quarter circle (again, it works similar for the lower quarter circle). By continuity, (2.1) *and* (2.2)*, we get*

$$u = C(u, v) = \frac{1}{2}\Big(C(u, u) + C(v, v)\Big),$$

which is equivalent to

$$C(u, u) + C(v, v) = 2u = u^2 + v^2.$$

This can be solved by

$$C(u, u) = u^2$$

for any $u \in \mathbb{I}$, resulting in

$$C(u, v) = \min\left(u, v, \frac{u^2 + v^2}{2}\right), \ \forall (u, v) \in \mathbb{I}^2.$$

3 Ordinal Sums

Definition 3.1 (see [4])
Let K be a (possibly) finite index set, $\{J_k\}_{k\in K}$ a partition of $\mathbb{I}$ with $J_k = [a_k, b_k]$, for $k \in K$, and $\{C_k\}_{k\in K}$ a collection of copulas. Then the ordinal sum of $\{C_k\}_{k\in K}$ w.r.t. $\{J_k\}_{k\in K}$ is defined by

$$C(u,v) = \begin{cases} a_k + (b_k - a_k)C_k\left(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k}\right), \text{ if } (u,v) \in J_k^2 \\ M(u,v), \text{ if } (u,v) \notin J_k^2. \end{cases}$$

One can easily verify that an ordinal sum is a copula: Let C be defined as above and $u = 0$, then either

$$C(0,v) = 0 + (b_k - 0)C_k\left(0, \frac{v-0}{b_k-0}\right)$$

or

$$C(0,v) = M(0,v) = 0.$$

The same holds, if $v = 0$. If $u = 1$, then either

$$\begin{aligned} C(1,v) &= a_k + (1-a_k)C_k\left(1, \frac{v-a_k}{1-a_k}\right) \\ &= a_k + (1-a_k)\frac{v-a_k}{1-a_k} \\ &= v \end{aligned}$$

or

$$C(1,v) = M(1,v) = 1$$

and, again, it works analogue for $v = 1$.

Theorem 3.2 (see [4])
Let C be a copula. Then C is an ordinal sum if and only if there exists a $t \in (0,1)$ sucht that $C(t,t) = t$.

Proof. "$\Rightarrow$" For any $k \in K$, take $t = a_k$ or $t = b_k$ to obtain

$$C(a_k, a_k) = a_k + (b_k - a_k)C_k(0,0) = a_k$$

or $C(b_k, b_k) = b_k$, respectively.
"$\Leftarrow$" Assume $\exists t \in (0,1)$ such that $C(t,t) = t$. Define for $(u,v) \in \mathbb{I}^2$

$$C_1(u,v) := \frac{C(tu, tv)}{t}$$

and

$$C_2(u,v) := \frac{C\Big(t+(1-t)u, t+(1-t)v\Big)}{1-t}.$$

Then C_1 and C_2 are copulas and C is the ordinal sum of $\{C_1, C_2\}$ w.r.t $\{[0,t],[t,1]\}$. $\square$

4 Shuffles of M

The term '*shuffles of M*' describes copulas that have a support consisting of line segments with slope -1 or 1. An informal way to construct this kind of copulas can be described as follows:

The mass distribution for a shuffle of M can be obtained by (1) placing the mass for M on $\mathbb{I}^2$, (2) cutting $\mathbb{I}^2$ vertically into a finite number of strips, (3) shuffling the strips with perhaps some of them flipped around their vertical axes of symmetry, and then (4) reassembling them to form the square again. The resulting mass distribution will correspond to a copula called a shuffle of M (see [2]).

Formally, let $n \in \mathbb{N}$, $\{J_i\}_{i=1,...,n}$ a partition of $\mathbb{I}$, π a permutation on $S_n = \{1, ..., n\}$ and ω a function with $\omega : S_n \to \{-1, 1\}$. The resulting shuffle of M is then denoted by

$$M\Big(n, \{J_i\}_{i=1,\dots,n}, \pi, \omega\Big)$$

(see [4]). If $\omega \equiv 1$, we call the resulting copula a *straight shuffle*, if $\omega \equiv -1$, we call it a *flipped shuffle*.
Note, that

$$W = M\Big(1, [0, 1], \mathrm{id}, -1\Big).$$

Shuffles of M offer some interesting properties that are worth mentioning. For this we first introduce the term *mutually completely dependent* which can be seen as the opposite of stochastic independence.

Definition 4.1 (see [4])
Let X, Y be two random variables. Then X and Y are called mutually completely dependent, if there exists a bijective function ϕ such that $\mathbb{P}(X = \phi(Y)) = 1$

Now, let the copula of some random variables X, Y be given by a shuffle of M. Then X and Y are mutually completely dependent, since the support of any shuffle of M is the graph of a bijective function (see [4]). The next theorem shows that we can use joint distribution functions of mutually completely dependent random variables to approximate a joint distribution of independent random variables with the same margin distributions arbitrarily closely. In other words, we can approximate any copula by a shuffle of M with an approximation error not greater than any $\epsilon > 0$.

Theorem 4.2 (see [4])
For any $\epsilon > 0$ and any copula C, there exists a shuffle of M, denoted by C_ϵ, such that

$$\sup_{u,v\in\mathbb{I}} |C_\epsilon(u, v) - C(u, v)| < \epsilon.$$

Proof. We proof this theorem for $C \equiv \Pi$, where Π is the product copula. The proof for an arbitrary C works similar, see [2].
Let $\epsilon > 0$ and $m \in \mathbb{N}$ such that $m \geq 4/\epsilon$. Following from Theorem 2.2.4 in [4] we have

$$|u_1, -v_1| < \frac{1}{m} \text{ and } |u_2, -v_2| < \frac{1}{m} \Rightarrow |C(u_1, u_2) - C(v_1, v_2)| < \frac{\epsilon}{2}.$$

Now, define $C_\epsilon = M(n, \{J_i\}_{i=1,\ldots,n}, \pi, \omega)$ where $n = m^2$, $\{J_i\}_{i=1,\ldots,n}$ is the partition of $\mathbb{I}$ into n subintervals with equal length, π is the permutation defined by $\pi(m(j-1)+k) = m(k-1)+j$ for $k, j = 1, ..., m$ and ω is arbitrary. Then $V_{C_\epsilon}\Big([0, p/m] \times [0, q/m]\Big) = V_\Pi\Big([0, p/m] \times [0, q/m]\Big) = pq/n$ for $p, q = 0, 1, ..., m$ which is equivalent to $C_\epsilon(p/m, q/m) = \Pi(p/m, q/m)$ for $p, q = 0, 1, ..., m$. Now, let $(u, v) \in \mathbb{I}^2$, then $\exists p, q \in \{0, 1, ..., m\}$ such that $|u - p/m| < 1/m$ and $|v - q/m| < 1/m$. Therefore, we have

$$\begin{aligned} |C_\epsilon(u,v) - \Pi(u,v)| &\leq |C_\epsilon(u,v) - C_\epsilon(p/m, q/m)| \\ &\quad + |C_\epsilon(p/m, q/m) - \Pi(p/m, q/m)| + |\Pi(p/m, q/m) - \Pi(u,v)| \\ &< \epsilon/2 + 0 + \epsilon/2 = \epsilon \end{aligned}$$

□

Theorem 4.3 (see [4])
Let C be a copula and suppose $C(a,b) = \theta$, where $(a,b) \in \mathbb{I}^2$ and θ satisfies

$$max(a+b-1, 0) \leq \theta \leq \min(a,b).$$

Then

$$C_L(u,v) \leq C(u,v) \leq C_U(u,v)$$

where

$$C_U = M\Big(4, \{[0,\theta], [\theta, a], [a, a+b-\theta], [a+b-\theta, 1]\}, (1,3,2,4), 1\Big)$$

and

$$C_L = M\Big(4, \{[0, a-\theta], [a-\theta, a], [a, 1-b+\theta], [1-b+\theta, 1]\}, (4,2,3,1), -1\Big)$$

Proof. C_U and C_L are explicitly given by

$$C_U(u,v) = \min\Bigg(u, v, \theta + (u-a)^+ + (v-b)^+\Bigg)$$

and

$$C_L(u,v) = \max\Bigg(0, u+v-1, \theta - (a-u)^+ + (b-v)^+\Bigg).$$

If $u \geq a$, then for $v \in \mathbb{I}$: $0 \leq C(u,v) - C(a,v) \leq u - a$ and if $u < a$, then $0 \leq C(a,v) - C(u,v) \leq a - u$, i.e.

$$-(a-u)^+ \leq C(u,v) - C(a,v) \leq (u-a)^+.$$

Doing the same with the second component gives

$$-(a-u)^{+}-(b-v)^{+} \leq C(u,v)-C(a,b) \leq (u-a)^{+}+(v-b)^{+}$$

for $u,v \in \mathbb{I}$. Since $C(a,b)=\theta$, we get

$$\theta-(a-u)^{+}-(b-v)^{+} \leq C(u,v) \leq \theta+(u-a)^{+}+(v-b)^{+}$$

which leads to

$$W(u,v) \leq C_L(u,v) \leq C(u,v) \leq C_U(u,v) \leq M(u,v).$$

□

5 Convex Sums

Definition 5.1 (see [4])
Let X be a continous random variable with distribution function F. Let C_x define a copula for any observation x of X. Then, the function defined by

$$C(u,v) = \int_{\mathbb{R}} C_x(u,v)dF(x)$$

is a called the convex sum of $\{C_x\}_{X=x}$ w.r.t. F, where F is called mixing distribution. If F has a parameter α, we write

$$C_\alpha(u,v) = \int_{\mathbb{R}} C_x(u,v)dF_\alpha(x).$$

The verification that convex sums are copulas is trivial: For $u, v, u_1, u_2, v_1, v_2 \in \mathbb{I}$ with $u_1 \le u_2$ and $v_1 \le v_2$ we have:

$$\begin{aligned}
&(1) \quad C(0,v) = \int_{\mathbb{R}} C_x(0,v)dF(x) = \int_{\mathbb{R}} 0dF(x) = 0 = C(u,0),\\
&(2) \quad C(1,v) = \int_{\mathbb{R}} C_x(1,v)dF(x) = \int_{\mathbb{R}} vdF(x) = v \cdot 1 = v,\\
&(3) \quad V_C([u_1,u_2] \times [v_1,v_2]) = \int_{\mathbb{R}} V_{C_x}([u_1,u_2] \times [v_1,v_2])dF(x) \ge 0.
\end{aligned}$$

Example 5.2 (see [4])
Let $\{C_x\}_{X=x}$ be a family of copulas defined by

$$C_x(u,v) = \begin{cases} M(u,v), \text{ if } |v-u| \ge x \\ W(u,v), \text{ if } |u+v-1| \ge 1-x \\ \dfrac{u+v-x}{2}, \text{ else} \end{cases}$$

for $x \in \mathbb{I}$. Let $F_\alpha(x) = x^\alpha$, $\alpha > 0$. Then C_α is given by

$$\begin{aligned}
C_\alpha(u,v) &= \int_{\mathbb{I}} C_x(u,v)dF_\alpha(x)\\
&= \int_{\mathbb{I}} C_x(u,v)\alpha x^{\alpha-1}dx\\
&= \int_0^{|v-u|} M(u,v)\alpha x^{\alpha-1}dx + \int_{1-|u+v-1|}^{1} W(u,v)\alpha x^{\alpha-1}dx + \int_{v-u}^{1-|u+v-1|} \frac{u+v-x}{2}\alpha x^{\alpha-1}dx\\
&= \min(u,v)|v-u|^\alpha + \max(u+v-1,0)\Big(1-\big(1-|u+v-1|\big)^\alpha\Big)\\
&+ \frac{u+v}{2}\Big(\big(1-|u+v-1|\big)^\alpha - |v-u|^\alpha\Big) - \frac{\alpha}{2(\alpha+1)}\Big(\big(1-|u+v-1|\big)^{\alpha+1} - |v-u|^{\alpha+1}\Big)
\end{aligned}$$

We now have 4 cases to consider:

1. $u \leq v$ *and* $u+v-1 \geq 0$, *i.e.* $M(u,v) = u$ *and* $W(u,v) = u+v-1$,
2. $u \leq v$ *and* $u+v-1 \leq 0$, *i.e.* $M(u,v) = u$ *and* $W(u,v) = 0$,
3. $u \geq v$ *and* $u+v-1 \geq 0$, *i.e.* $M(u,v) = v$ *and* $W(u,v) = u+v-1$ *and*
4. $u \geq v$ *and* $u+v-1 \leq 0$, *i.e.* $M(u,v) = v$ *and* $W(u,v) = 0$.

For the first case we can calculate

$$\begin{aligned}
C_\alpha(u,v) &= u|v-u|^\alpha - \frac{u+v}{2}|v-u|^\alpha + (u+v-1) - (u+v-1)\Big(1-|u+v-1|\Big)^\alpha \\
&+ \frac{u+v}{2}\Big(1-|u+v-1|\Big)^\alpha - \frac{\alpha}{2(\alpha+1)}\left(\Big(1-|u+v-1|\Big)^{\alpha+1} - |v-u|^{\alpha+1}\right) \\
&= \frac{u-v}{2}|v-u|^\alpha + (u+v-1) + \frac{2-u-v}{2}\Big(1-|u+v-1|\Big)^\alpha \\
&- \frac{\alpha}{2(\alpha+1)}\left(\Big(1-|u+v-1|\Big)^{\alpha+1} - |v-u|^{\alpha+1}\right) \\
&= -\frac{|v-u|^{\alpha+1}}{2} + (u+v-1) + \frac{(1-|u+v-1|)^{\alpha+1}}{2} \\
&- \frac{\alpha}{2(\alpha+1)}\left(\Big(1-|u+v-1|\Big)^{\alpha+1} - |v-u|^{\alpha+1}\right) \\
&= (u+v-1) + \left(\frac{1}{2} - \frac{\alpha}{2(\alpha+1)}\right)\left(\Big(1-|u+v-1|\Big)^{\alpha+1} - |v-u|^{\alpha+1}\right) \\
&= W(u,v) + \frac{1}{2(\alpha+1)}\left(\Big(1-|u+v-1|\Big)^{\alpha+1} - |v-u|^{\alpha+1}\right).
\end{aligned}$$

The other cases are left out since they give the same result.

6 Copulas with Prescribed Horizontal and Vertical Sections

First, let us recall the definition of horizontal and vertical sections:

Definition 6.1 (Horizontal and vertical sections (see [4]))
Let C be a copula and let $a \in \mathbb{I}$. Then the function $C(\cdot, a) : \mathbb{I} \to \mathbb{I}$, $t \mapsto C(t, a)$ is called the horizontal section of C in a and $C(a, \cdot) : \mathbb{I} \to \mathbb{I}$, $t \mapsto C(a, t)$ is called the vertical section of C in a.

6.1 Copulas with Linear Sections

We start with a rather trivial case which is that we want to construct a copula C that has a linear horizontal section. That is, for any $(u, v) \in \mathbb{I}^2$ we have

$$C(u, v) = a(v)u + b(v).$$

From the boundary conditions we get

$$0 = C(0, v) = b(v) \quad \Rightarrow \quad v = C(1, v) = a(v),$$

which results in $C(u, v) = uv$. Since this holds also for the vertical section, the only copula with linear sections is the product copula Π (see [4]).

6.2 Copulas with Quadratic Sections

Let C be a copular an suppose it has a quadratic horizontal section, then for any $(u, v) \in \mathbb{I}^2$ we have

$$C(u, v) = a(v)u^2 + b(v)u + c(v).$$

Again, from boundary conditions we get

$$0 = C(0, v) = c(v) \quad \Rightarrow \quad v = C(1, v) = a(v) + b(v).$$

Now, choose a function ψ such that

$$\psi(v) = -a(v) \quad \Rightarrow \quad b(v) = v - a(v) = v + \psi(v).$$

This results in

$$C(u, v) = -\psi(v)u^2 + \Big(v + \psi(v)\Big)u = uv + \psi(v)(1 - u)u, \tag{6.1}$$

where ψ must be chosen such that C is 2-increasing and $\psi(1) = \psi(0) = 0$ to satisfy a copula's boundary conditions (see [4]).

Example 6.2 (see [4])
We want to construct a symmetric copula C with quadratic sections in both u and v. As a consequence we have

$$\psi(v) := \theta v(1 - v)$$

for some constant θ. Now

$$C_\theta(u, v) = uv + \theta v(1 - v)u(1 - u),$$

where. Then the boundary conditions for a copula are satisfied and for a rectangle $[u_1, u_2] \times [v_1, v_2] \in \mathbb{I}^2$ we have

$$V_{C_\theta} = \text{ [...] } = (u_2 - u_1)(v_2 - v_1)\Big(1 + \theta(1 - u_1 - u_2)(1 - v_1 - v_2)\Big),$$

which is greater or equal than 0 for every $(u, v) \in \mathbb{I}^2$ if and only if $\theta \in [-1, 1]$. This family of copulas is called Farlie-Gumbel-Morgenstern family and contains all copulas with quadratic sections in both u and v.

The next reasonable question would be how to generally choose ψ, aside from satisfying the 2-increasing-criterium and the boundary conditions, in order for (6.1) to be a copula. An answer to that is given by the following theorem and corollary from [5].

Theorem 6.3 (see [4])
Let ψ be a function with domain $\mathbb{I}$ and let C be given by

$$C(u, v) = uv + \psi(v)u(1 - u).$$

Then C is a copula if and only if

1. $\psi(0) = \psi(1) = 0$
2. *ψ satisfies the Lipschitz condition, i.e. for all $v_1, v_2 \in \mathbb{I}$*

$$|\psi(v_2) - \psi(v_1)| \leq |v_2 - v_1|.$$

Furthermore, C is absolutely continous.

Proof. The first condition is equivalent to

$$C(0, v) = C(u, 0) = 0 \quad \text{and} \quad C(1, v) = v, \;\; C(u, 1) = u$$

Additionally, C is 2-increasing if and only if for all $u_1, u_2, v_1, v_2 \in \mathbb{I}$, $u_1 \leq u_2$, $v_1 \leq v_2$, it holds

$$\begin{aligned}
V_C([u_1, u_2] \times [v_1, v_2]) &= u_2 v_2 + \psi(v_2) u_2(1 - u_2) - u_1 v_2 - \psi(v_2) u_1 (1 - u_1) \\
&\quad - u_2 v_1 - \psi(v_1) u_2 (1 - u_2) + u_1 v_1 + \psi(v_1) u_1 (1 - u_1) \\
&= (u_2 - u_1)(v_2 - v_1) + \Big(\psi(v_2) - \psi(v_1)\Big)\Big(u_2(1 - u_2) - u_1(1 - u_1)\Big) \\
&= (u_2 - u_1)(v_2 - v_1) + \Big(\psi(v_2) - \psi(v_1)\Big)\Big((u_2 - u_1)(1 - u_1 - u_2)\Big) \\
&= (u_2 - u_1)\Bigg((v_2 - v_1) + \Big(\psi(v_2) - \psi(v_1)\Big)(1 - u_1 - u_2)\Bigg) \\
&\geq 0
\end{aligned}$$

If $u_1 = u_2$ or $v_1 = v_2$, then $V_C([u_1, u_2] \times [v_1, v_2]) = 0$ and if $u_1 + u_2 = 1$, then $V_C([u_1, u_2] \times [v_1, v_2]) = V_\Pi([u_1, u_2] \times [v_1, v_2]) \geq 0$, since the product copula Π is 2-increasing. So, assume $u_1 < u_2$ and $v_1 < v_2$. Then it has to hold

$$\frac{\psi(v_2) - \psi(v_1)}{v_2 - v_1} \leq \frac{1}{u_2 + u_1 - 1},$$

if $u_1 + u_2 > 1$ and

$$\frac{\psi(v_2) - \psi(v_1)}{v_2 - v_1} \geq \frac{1}{u_2 + u_1 - 1},$$

if $u_1 + u_2 < 1$. Since $\inf\Big\{1/(u_1 + u_2 - 1) : 0 \leq u_1 \leq u_2 \leq 1,\ u_1 + u_2 > 1\Big\} = 1$ and $\sup\Big\{1/(u_1 + u_2 - 1) : 0 \leq u_1 \leq u_2 \leq 1,\ u_1 + u_2 < 1\Big\} = -1$, C is 2-increasing if and only if

$$-1 \leq \frac{\psi(v_2) - \psi(v_1)}{v_2 - v_1} \leq 1,$$

which is equivalent to the second condition and implies absolute continuity of ψ, which in return implies the absolute continuity of C.

□

Corollary 6.4
The function C as in (6.1) *is a copula if and only if the following are satisfied:*

1. *ψ is absolutely continous on $\mathbb{I}$.*
2. *$|\psi'(v)| > 1$ only for a finite number of $v \in \mathbb{I}$.*
3. *$|\psi(v)| \leq M(v, 1 - v)$ for all $v \in \mathbb{I}$.*

6.3 Copulas with Cubic Sections

Since the construction of copulas with cubic sections is just an extended version of the construction of quadratic sections, the results will not be proven, but, for those who are interested, can be looked up in [3].

If C is a copula with cubic horizontal section, then

$$C(u,v) = a(v)u^3 + b(v)u^2 + c(v)u + d(v)$$

and, once again, with boundary conditions we get

$$d(v) = 0 \;\; \Rightarrow \;\; c(v) = v - a(v) - b(v).$$

Let $\alpha(v) = -a(v) - b(v)$ and $\beta(v) = -2a(v) - b(v)$ with $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$, then

$$C(u,v) = uv + u(1-u)\Big(\alpha(v)(1-u) + \beta(v)u\Big)$$

(see [4]).

Again, the question arises how to choose α and β and it will be answered by the next theorem.

Theorem 6.5 (see [4])
Let α, β be two functions with domain $\mathbb{I}$ satisfying $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$ and let C be given by

$$C(u,v) = uv + u(1-u)\Big(\alpha(v)(1-u) + \beta(v)u\Big).$$

Then C is a copula, if and only if

1. *α and β are absolutely continous.*
2. *For almost all $v \in \mathbb{I}$, either*

$$-1 \leq \alpha'(v) \leq 2 \text{ and } -2 \leq \beta'(v) \leq 1$$

or

$$\Big(\alpha'(v)\Big)^2 - \alpha'(v)\beta'(v) + \Big(\beta'(v)\Big)^2 - 3\alpha'(v) + 3\beta'(v) \leq 0.$$

Furthermore, C is absolutely continous.

We now want to find *all* copulas that have both cubic horizontal sections and vertical sections. That is, copulas satisfying:

$$C(u,v) = uv + u(1-u)\Big(\alpha(v)(1-u) + \beta(v)u\Big) \tag{6.2}$$

and

$$C(u,v) = uv + v(1-v)\Big(\gamma(u)(1-v) + \epsilon(u)v\Big). \tag{6.3}$$

with γ,ϵ satisfying the same conditions as α, β.

Theorem 6.6 (see [4])
Suppose that a copula C has cubic sections in both u and v, i.e. C is given by both (6.2) *and* (6.3)*. Then*

$$C(u,v) = uv + uv(1-u)(1-v)\Big(A_1 v(1-u) + A_2(1-v)(1-u) + B_1 uv + B_2 u(1-v)\Big),$$

where $A_1, A_2, B_1, B_2 \in \mathbb{R}$ *such that for all* $(x,y) \in \Big\{(A_2,A_1),(B_1,B_2),(B_1,A_1),(A_2,B_2)\Big\}$

$$-1 \leq x \leq 2 \text{ and } -2 \leq y \leq 1$$

or

$$x^2 - xy + y^2 - 3x + 3y \leq 0.$$

With this result we can find an explicit expression of α, β, γ and ϵ:
Using (6.2) we get

$$\Big(\alpha(v)(1-u) + \beta(v)u\Big) = v(1-v)\Big(A_1 v(1-u) + A_2(1-v)(1-u) + B_1 uv + B_2 u(1-v)\Big)$$

$$\Leftrightarrow$$

$$\Big(\alpha(v)(1-u) + \beta(v)u\Big) = v(1-v)\bigg(\Big(A_1 v + A_2(1-v)\Big)(1-u) + \Big(B_1 v + B_2(1-v)\Big)u\bigg)$$

which gives

$$\begin{aligned}\alpha(v) &= v(1-v)\Big(A_1 v + A_2(1-v)\Big),\\ \beta(v) &= v(1-v)\Big(B_1 v + B_2(1-v)\Big)\end{aligned}$$

and similarily for γ and ϵ

$$\begin{aligned}\gamma(u) &= u(1-u)\Big(B_2 u + A_2(1-u)\Big),\\ \epsilon(u) &= u(1-u)\Big(B_1 u + A_1(1-u)\Big)\end{aligned}$$

(see [4]).

Example 6.7 (see [4])
Let a, b *be constants such that* $b \in [-1,2]$, $|a| \leq b+1$ *for each* $b \in [-1,1/2]$ *and* $|a| \leq (6b - 3b^2)^{1/2}$ *for each* $b \in [1/2,2]$. *Now, set* $A_1 = B_2 = a - b$ *and* $A_2 = B_1 = a + b$ *which satisfy the previos theorem's conditions. Then, we have*

$$\begin{aligned}\alpha(v) &= v(1-v)(a + b - 2bv),\\ \beta(v) &= v(1-v)(a - b + 2bv)\end{aligned}$$

and $\gamma \equiv \alpha$, $\epsilon \equiv \beta$. *The resulting copula is given by*

$$\begin{aligned}C_{a,b}(u,v) &= uv + u(1-u)\Big(\alpha(v)(1-u) + \beta(v)u\Big)\\ &= uv + uv(1-u)(1-v)\Big(a + b(1-2v)(1-u) + b(2v-1)u\Big)\\ &= uv + uv(1-u)(1-v)\Big(a + b(1-2u)(1-2v)\Big).\end{aligned}$$

Note that $C_{a,0}$ *gives the Farlie-Gumbel-Morgenstern family from the previous section.*

7 Copulas with Prescribed Diagonal Sections

Before we start, we recall the definition of diagonal sections and the dual of a copula:

Definition 7.1 (see [4])
Let C be a copula. Then $\delta_C : \mathbb{I} \to \mathbb{I}$, $\delta_C(t) := C(t,t)$ is called the diagonal section of C and the dual of C is given by the function

$$\tilde{\delta}_C : \mathbb{I} \to \mathbb{I} \;\; \tilde{\delta}_C(t) = 2t - \delta_C(t).$$

Diagonal sections have an interesting property. We can explicitly give the distribution function of the order statistics $\max(X,Y)$ and $\min(X,Y)$ where X and Y are random variables having a common distribution F and copula C (see [4]): following from Sklar's theorem, we have

$$\mathbb{P}\Big(\max(X,Y) \leq t\Big) = \mathbb{P}(X \leq t, Y \leq t) = C\Big(F(t), F(t)\Big) = \delta_C\Big(F(t)\Big)$$

and

$$\begin{aligned}\mathbb{P}\Big(\min(X,Y) \leq t\Big) &= \mathbb{P}(X \leq t) + \mathbb{P}(Y \leq t) - \mathbb{P}(X \leq t, Y \leq t)\\ &= 2F(t) - \delta_C\Big(F(t)\Big) = \tilde{\delta}_C\Big(F(t)\Big).\end{aligned}$$

Before we move on, let us characterize what we mean when we simply speak of *diagonals*:

Definition 7.2 (see [4])
A function $\delta : \mathbb{I} \to \mathbb{I}$ is called a diagonal if it satisfies the following three properties:

(1) $\delta(1) = 1$,

(2) $0 \leq \delta(t_2) - \delta(t_1) \leq 2(t_2 - t_1)$, *for any* $t_1, t_2 \in \mathbb{I}$, $t_1 \leq t_2$,

(3) $\delta(t) \leq t$ *for any* $t \in \mathbb{I}$.

With every diagonal we can construct a so called *diagonal copula*:

Theorem 7.3 (see [4])
Let δ be any diagonal and set

$$C(u,v) := \min\left(u, v, \frac{1}{2}\Big(\delta(u) + \delta(v)\Big)\right).$$

Then C is a copula whose diagonal section is δ.

The proof of this theorem is rather technical and left out here, but it can be found in [1]. We conclude this thesis by a theorem that can give us the joint distribution function of the order statistics.

Theorem 7.4 (see [4])
Suppose X and Y are continous random variables with copula C and a common marginal distribution. Then the joint distribution function of $\min(X,Y)$ and $\max(X,Y)$ is the Fréchet-Hoeffding upper bound M if and only if C is a diagonal copula.

Proof. Using Sklar's theorem, we can assume that X and Y have the copula C as joint distribution function and then show the equivalence of C being a diagonal copula and the joint distribution function of $\min(X,Y)$ and $\max(X,Y)$ being M. Let $H(z,\tilde{z})$ be the joint distribution of $Z = \max(X,Y)$ and $\tilde{Z} = \min(X,Y)$. As we have shown above, the distributions of Z and $\tilde{Z}$ are the diagonal section δ_C and the dual of C $\tilde{\delta}_C$, respectively. Setting $\delta = \delta_C$ gives

$$H(z,\tilde{z}) = P\Big(\max(X,Y) \leq z, \min(X,Y) \leq \tilde{z}\Big) = \begin{cases} \delta(z), & \text{if } z \leq \tilde{z}, \\ C(z,\tilde{z}) + C(\tilde{z},z) - \delta(\tilde{z}), & \text{if } z \geq \tilde{z}. \end{cases}$$

"$\Leftarrow$" Assume C is a diagonal copula. Then, if $z \geq \tilde{z}$

$$H(z,\tilde{z}) = 2C(z,\tilde{z}) - \delta(\tilde{z}) = \min\Big(2\tilde{z} - \delta(\tilde{z}), \delta(z)\Big) = \min\Big(\tilde{\delta}(\tilde{z}), \delta(z)\Big).$$

If $z < \tilde{z}$, then $\delta(z) = \min\Big(\tilde{\delta}(\tilde{z}), \delta(z)\Big)$ since $\delta(z) \leq \delta(\tilde{z}) \leq \tilde{z} \leq \tilde{\delta}(\tilde{z})$. This gives $H(z,\tilde{z}) = M\Big(\delta(z), \tilde{\delta}(\tilde{z})\Big)$.
"$\Rightarrow$" Assume $H(z,\tilde{z}) = M\Big(\delta(z), \tilde{\delta}(\tilde{z})\Big)$. Here, we assume C to be symmetric (for a general proof see [1]). If $z > \tilde{z}$, then

$$2C(z,\tilde{z}) - \delta(\tilde{z}) = M\Big(\delta(z), \tilde{\delta}(\tilde{z})\Big) = \min\Big(2\tilde{z} - \delta(\tilde{z}), \delta(z)\Big)$$

and thus

$$C(z,\tilde{z}) = \min\left(\tilde{z}, \frac{1}{2}\Big(\delta(z) + \delta(\tilde{z})\Big)\right).$$

Since we assumed C to be symmetric, we have

$$C(z,\tilde{z}) = \min\left(z, \frac{1}{2}\Big(\delta(z) + \delta(\tilde{z})\Big)\right)$$

for $z \leq \tilde{z}$, which completes the proof. □

Bibliography

[1] Beneŝ V, Ŝtêpán J: *Distributions with given marginals and moment problems.* Kluwer, 1997.

[2] Mikusiński, P, Sherwood, H, and Taylor, MD: *Probabilistic interpretations of copulas and their convex sums.* In *Advances in probability distributions with given marginals*, pages 95–112. Springer, 1991.

[3] Nelsen, Roger B: *Dependence and order in families of archimedean copulas.* Journal of Multivariate Analysis, 60(1):111–122, 1997.

[4] Nelsen, Roger B: *An introduction to copulas.* Springer Science & Business Media, 2007.

[5] Quesada-Molina, JJ and Rodríguez-Lallena, JA: *Bivariate copulas with quadratic sections.* Journaltitle of Nonparametric Statistics, 5(4):323–337, 1995.