Homework assignment #1 for Random Fields I

Due Thursday, May 07, 2009

1. Prove the existence of a random field with the following finite dimensional distributions and specify the measurable spaces \((E_{t_1...t_n}, E_{t_1...t_n})\).
   a) The finite dimensional distributions are multivariate Gaussian.
   b) The marginals are Poisson and \(X_t, X_s\) are independent for \(t \neq s\).

2. Give an example for a family of probability measures \(\{P_{t_1...t_n}\}\) which do not fulfil the conditions in the theorem of Kolmogorov.

3. Prove Proposition 1.1.2: The family of measures \(\{P_{t_1...t_n}\text{ on } (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\}, n \geq 1, t_1, \ldots, t_n \in T\) satisfies the conditions in the theorem of Kolmogorov (i.e. symmetry and consistency) if and only if \(\forall n \geq 2, \forall s = (s_1, \ldots, s_n)^\top \in \mathbb{R}^n, \) the following two conditions hold:
   - \(\varphi_{P_{t_1...t_n}}((s_1, \ldots, s_n)^\top) = \varphi_{P_{t_1...t_n}}((s_{i_1}, \ldots, s_{i_n})^\top)\) for any permutation \((i_1, \ldots, i_n)\) of \((1, \ldots, n)\),
   - \(\varphi_{P_{t_1...t_{n-1}}}((s_1, \ldots, s_{n-1})^\top) = \varphi_{P_{t_1...t_n}}((s_1, \ldots, s_{n-1}, 0)^\top)\).

4. Give an example (not the one presented in the lecture) of a non-continuous random function which has a continuous modification.

5. Let the probability space \((\Omega, \mathcal{F}, P)\) be given by the unit interval \(\Omega = [0, 1]\), the Borel sigma algebra \(\mathcal{F} = \mathcal{B}([0,1])\) and the Lebesgue measure \(P = \lambda\). The binary expansion of \(\omega \in \Omega\) is given by \(\omega = \sum_{k=1}^{\infty} a_k(\omega) 2^{-k}\) with coefficients \(a_1(\omega), a_2(\omega), \ldots \in \{0, 1\}\). In cases where this representation is not unique (for example \(0.5 = 1 \cdot 2^{-1} + \sum_{k=2}^{\infty} 0 \cdot 2^{-k} = 0 \cdot 2^{-1} + \sum_{k=2}^{\infty} 1 \cdot 2^{-k}\)) we use the ”shorter” variant filled up with zeros.
   a) Prove that for each \(k \in \mathbb{N}, a_k\) is Bernoulli distributed with parameter 0.5, i.e. \(P(a_k = 0) = P(a_k = 1) = 0.5\).
   b) Prove that the random variable \(\xi\) defined by \(\xi(\omega) = \sum_{k=1}^{\infty} a_k(\omega) 2^{-k}\) is uniformly distributed on the unit interval \([0, 1]\).