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Homework assignment #1 for Random Fields I

Due Thursday, May 07, 2009

1. Prove the existence of a random field with the following finite dimensional distributions and specify the measurable spaces $(E_{t_1 \ldots t_n}, \mathcal{E}_{t_1 \ldots, t_n})$.
   a) The finite dimensional distributions are multivariate Gaussian.
   b) The marginals are Poisson and X_t, X_s are independent for $t \neq s$.
2. Give an example for a family of probability measures $\{P_{t_1 \ldots t_n}\}$ which do not fulfil the conditions in the theorem of Kolmogorov.
3. Prove Proposition 1.1.2: The family of measures $\{P_{t_1 \ldots t_n}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n,)), n \geq 1$, $t_1, \ldots, t_n \in T\}$ satisfies the conditions in the theorem of Kolmogorov (i.e. symmetry and consistency) if and only if $\forall n \geq 2$, $\forall s = (s_1, \ldots, s_n)^\top \in \mathbb{R}^n$, the following two conditions hold:
   - $\varphi_{P_{t_1 \ldots t_n}}\left((s_1, \ldots, s_n)^\top\right) = \varphi_{P_{t_{i_1} \ldots t_{i_n}}}\left((s_{i_1}, \ldots, s_{i_n})^\top\right)$ for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$,
   - $\varphi_{P_{t_1 \ldots t_{n-1}}}\left((s_1, \ldots, s_{n-1})^\top\right) = \varphi_{P_{t_1 \ldots t_n}}\left((s_1, \ldots, s_{n-1}, 0)^\top\right)$.
4. Give an example (not the one presented in the lecture) of a non-continuous random function which has a continuous modification.
5. Let the probability space $(\Omega, \mathcal{F}, P)$ be given by the unit interval $\Omega = [0,1]$, the Borel sigma algebra $\mathcal{F} = \mathcal{B}([0,1])$ and the Lebesgue measure $P = \lambda$. The binary expansion of $\omega \in \Omega$ is given by $\omega = \sum_{k=1}^{\infty} a_k(\omega) 2^{-k}$ with coefficients $a_1(\omega), a_2(\omega), \ldots \in \{0,1\}$. In cases where this representation is not unique (for example $0.5 = 1 \cdot 2^{-1} + \sum_{k=2}^{\infty} 0 \cdot 2^{-k} = 0 \cdot 2^{-1} + \sum_{k=2}^{\infty} 1 \cdot 2^{-k}$) we use the "shorter" variant filled up with zeros.
   a) Prove that for each $k \in \mathbb{N}$, a_k is Bernoulli distributed with parameter 0.5, i.e. $\mathbb{P}(a_k = 0) = \mathbb{P}(a_k = 1) = 0.5$.
   b) Prove that the random variable ξ defined by $\xi(\omega) = \sum_{k=1}^{\infty} a_k(\omega) 2^{-k}$ is uniformly distributed on the unit interval $[0,1]$.