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Homework assignment #2 for Random Fields I

Due Thursday, May 14, 2009

1. The generalized inverse function $F^{-1} : [0,1] \to \mathbb{R}$ of a cumulative distribution function (cdf) F is defined by

$$F^{-1}(y) = \inf \{x \in \mathbb{R} : F(x) \geq y\}.$$

Let $Y \sim U([0,1])$ and define the random variable X by $X(\omega) = F^{-1}(Y(\omega))$. Prove that the cumulative distribution function of X equals F.

Note: This procedure is called 'inverse transform sampling' and can be used to simulate random variables with cdf F. With the help of exercise 5 on sheet 1 a standard uniformly distributed random variable can be simulated by a Bernoulli distributed random variable, e.g. by tossing a coin.

2. For a random field $X = \{X_t, t \in T\}$ its expectation function $m : T \to \mathbb{R}$ is defined by $m_t = \mathbb{E}X_t$ for all $t \in T$ (if $\mathbb{E}X_t$ exists). The (non-centered) covariance function $C : T \times T \to \mathbb{R}$ is defined by $C(s,t) = \mathbb{E}(X_s X_t)$ for all $s,t \in T$ (if it exists). Prove that a Gaussian random field is uniquely determined by its mean and covariance function.

3. A random process W defined on $\mathcal{B}(\mathbb{R}^d)$ such that, for all sets $A, B \in \mathcal{B}(\mathbb{R}^d)$ with $|A|$ and $|B|$ finite,
   - $W(B) \sim \mathcal{N}(0, |B|)$,
   - $A \cap B = \emptyset \Rightarrow W(A \cup B) = W(A) + W(B)$ almost surely,
   - $A \cap B = \emptyset \Rightarrow W(A)$ and $W(B)$ are independent,

   is called white noise indexed by Borel sets. Define the field $X = \left\{X_t, t \in [0,\infty)^d\right\}$ by $X_t = W([0,t])$ where $[0,t]$ is the paraxial rectangle with lower left corner 0 and upper right corner t. Show (for $d = 2$) that X is a Gaussian random field with $\mathbb{E}X_t = 0$ and $\operatorname{Cov}(X_s, X_t) = \prod_{j=1}^{d} \min\{s_j, t_j\}$.

   Note: The process X is called Brownian sheet or multiparameter Brownian motion. For $d = 1$ it is the Brownian motion.

4. Let $\eta_1, \eta_2, \ldots$ non-negative i.i.d. random variables and $X_1, X_2, \ldots$ non-negative i.i.d. random variables, where η_j and X_j are independent for all j. Define the process $N = \{N_t, t \in [0, \infty)\}$ by

$$N_t(\omega) = \sup\left\{n \in \mathbb{N} : \sum_{j=1}^{n} \eta_j(\omega) \le t\right\}.$$

The process $Y = \{Y_t, t \in [0, \infty)\}$ is given by

$$Y_t(\omega) = y_0 + ct - \sum_{j=1}^{N_t(\omega)} X_j(\omega), t \ge 0,$$

with positive constants y_0 and c. Draw a sketch of a trajectory of Y.

Note: The process Y_t is the classical Cramér-Lundberg model in insurance mathematics. It describes the surplus at time t of an insurance portfolio, where y_0 is the initial capital, c is the premium intensity and $(X_j)_{j\ge1}$ is the sequence of claim sizes. N_t is called the claim number process.

5. Let Φ be a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$.
   a) Write down the finite dimensional distributions of Φ for disjoint bounded Borel sets $B_1, \ldots, B_n$.
   b) Compute the expectation $\mathbb{E}\Phi(B)$ and the variance $\operatorname{Var}\Phi(B)$ for bounded Borel sets B and interpret why λ is called the intensity of Φ.