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Homework assignment #3 for Random Fields I

Due Thursday, May 28, 2009

1. Let Φ be a homogeneous Poisson process in $\mathbb{R}^d$ of intensity $\lambda > 0$.
   a) *Independent thinning:* Each point of Φ is retained with probability $p \in (0,1)$ and deleted with probability $1-p$, independently of other points. Prove that the thinned process $\widetilde{\Phi}$ is a homogeneous Poisson process of intensity λp.
   b) *Superposition:* The superposition of two independent Poisson processes Φ_1 with intensity λ_1 and Φ_2 with intensity λ_2 is the union $\widetilde{\Phi} = \Phi_1 \cup \Phi_2$. Prove that $\widetilde{\Phi}$ is a homogeneous Poisson process with intensity $\lambda_1 + \lambda_2$.

   Hint: The void probabilities of a point process are defined by $v_B = P(\Phi(B) = 0)$ for all Borel sets B. A simple point process is characterized by the void probabilities v_B as B ranges through the Borel sets.

2. Consider the Shot-Noise-Field $X_t = \sum_{v\in\Phi} g(t-v)$ where Φ is a homogeneous Poisson process of intensity λ and $g : \mathbb{R}^d \to \mathbb{R}$ is a deterministic function fulfilling some integrability conditions. Prove that
   a) $\mathbb{E}X_t = \lambda \int_{\mathbb{R}^d} g(z)\, dz$
   b) $\operatorname{Cov}(X_t, X_s) = \lambda \int_{\mathbb{R}^d} g(t-s-z)\, g(z)\, dz$.

   Hint: Campbell's theorem can be useful: Let Φ be a homogeneous Poisson process in $\mathbb{R}^d$ with intensity λ and $f : \mathbb{R}^d \to \mathbb{R}$ a non-negative measurable function. Then it holds that
   $$\mathbb{E}\sum_{v\in\Phi} f(v) = \lambda \int_{\mathbb{R}^d} f(z)\, dz.$$
   Further, it holds that
   $$\mathbb{E}\sum_{\substack{v,w\in\Phi\\ v\neq w}} f(v)\, g(w) = \lambda^2 \int_{\mathbb{R}^d} f(z)\, dz \int_{\mathbb{R}^d} g(z)\, dz$$

3. Consider the Ising model given in the lecture.
   a) Prove that the Ising model has the Markov property with respect to the 4-neighbourhood relation $\sim_4$. Does the Ising model possess the Markov property with respect to the 8-neighbourhood relation $\sim_8$?

b) For any $\Gamma \subset T_N$ the joint distribution of the system of random variables $\{X_t, t \in \Gamma\}$ is denoted by $P^{\Gamma}_{T_N}$, i.e. for $\Gamma = \{t_1, \ldots, t_n\}$ and $x_{t_1}, \ldots, x_{t_n} = \pm 1$ it holds that $P^{\Gamma}_{T_N}(x_{t_1}, \ldots, x_{t_n}) = P(X_{t_1} = x_{t_1}, \ldots, X_{t_n} = x_{t_n})$. For any such Γ the expectation $\sigma_\Gamma = \mathbb{E}\prod_{t\in\Gamma} X_t$ is called correlation function.

Prove that the probabilities $P^{\Gamma}_{T_N}$ can be expressed in terms of the correlation functions σ_Γ as follows:

$$P^{\Gamma}_{T_N}(x_{t_1}, \ldots, x_{t_n}) = \frac{(-1)^k}{2^n} \sum_{\Gamma' \subset \Gamma} C_{\Gamma'} \sigma_{\Gamma'}$$

where k is the number of values x_{t_i} equal to -1 and $C_{\Gamma'} = \prod_{t\in\Gamma\setminus\Gamma'} x_t$.

c) Assume that $J = 0$, i.e. there is no interaction between the particles. Prove that for any $\Gamma \subset T_N$ the correlation function σ_Γ is given by

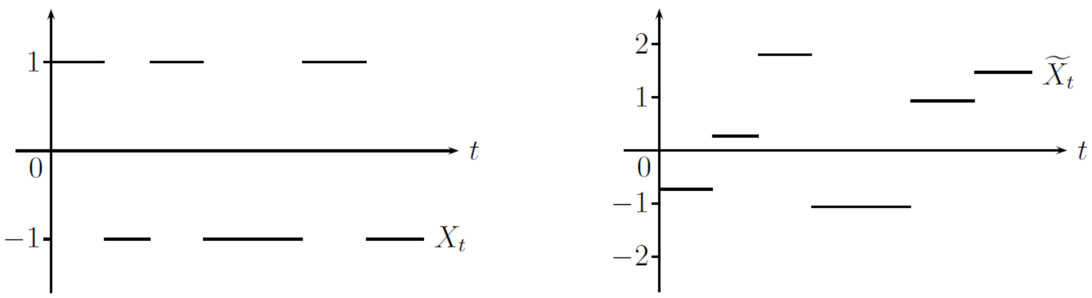
$$\sigma_\Gamma = \left(\frac{\exp\left(-\frac{mB}{K_0 t_0}\right) - \exp\left(\frac{mB}{K_0 t_0}\right)}{\exp\left(-\frac{mB}{K_0 t_0}\right) + \exp\left(\frac{mB}{K_0 t_0}\right)} \right)^{|\Gamma|}.$$

4. Prove that the covariance function $C(t,s) = \mathbb{E}X_t X_s - \mathbb{E}X_t \mathbb{E}X_s$ of a random field X is positive semi-definite, i.e. $\forall n \in \mathbb{N}$, $\forall t_1, \ldots, t_n \in T$, $\forall z_1, \ldots, z_n \in \mathbb{R}$ it holds that
$$\sum_{i=1}^{n}\sum_{j=1}^{n} C(t_i, t_j)\, z_i z_j \geq 0.$$

5. The process $X = \{X_t, t \in [0,\infty)\}$ is defined as follows: X_0 takes values $+1$ and -1 with probability $1/2$ each and $X_t = X_0 \cdot (-1)^{Y([0,t])}$ for $t > 0$ where Y is a homogeneous Poisson process on $[0,\infty)$ with intensity $\lambda > 0$, independent of X_0. See Figure 1 (a) for a trajectory of X.

a) Compute the expectation function of X and its covariance function.

b) Let's assume that the process X takes not only values ± 1 but when the kth point of Y occurs, it assumes a random value V_k and retains that value till the next point of Y occurs, i.e. $X_0 = V_0$ and $X_t = V_{Y([0,t])}$ for $t > 0$ where $V_0, V_1, \ldots$ are i.i.d. random variables with $V_0 \sim \mathcal{N}(0, \sigma^2)$. Figure 1 (b) shows a trajectory of $\widetilde{X}$. Specify the marginal distribution. Is it a Gaussian process?



(a) Trajectory of the process X in Exercise 5 a).

(b) Trajectory of the process $\widetilde{X}$ in Exercise 5 b).

Figure 1: Illustration for Exercise 5.