Homework assignment #3 for Random Fields I

Due Thursday, May 28, 2009

1. Let $\Phi$ be a homogeneous Poisson process in $\mathbb{R}^d$ of intensity $\lambda > 0$.
   a) Independent thinning: Each point of $\Phi$ is retained with probability $p \in (0, 1)$ and deleted with probability $1 - p$, independently of other points. Prove that the thinned process $\tilde{\Phi}$ is a homogeneous Poisson process of intensity $\lambda p$.
   b) Superposition: The superposition of two independent Poisson processes $\Phi_1$ with intensity $\lambda_1$ and $\Phi_2$ with intensity $\lambda_2$ is the union $\tilde{\Phi} = \Phi_1 \cup \Phi_2$. Prove that $\tilde{\Phi}$ is a homogeneous Poisson process with intensity $\lambda_1 + \lambda_2$.

   Hint: The void probabilities of a point process are defined by $v_B = P(\Phi(B) = 0)$ for all Borel sets $B$. A simple point process is characterized by the void probabilities $v_B$ as $B$ ranges through the Borel sets.

2. Consider the Shot-Noise-Field $X_t = \sum_{v \in \Phi} g(t - v)$ where $\Phi$ is a homogeneous Poisson process of intensity $\lambda$ and $g : \mathbb{R}^d \to \mathbb{R}$ is a deterministic function fulfilling some integrability conditions. Prove that
   a) $\mathbb{E}X_t = \lambda \int_{\mathbb{R}^d} g(z) \, dz$
   b) $\text{Cov}(X_t, X_s) = \lambda \int_{\mathbb{R}^d} g(t - s - z) g(z) \, dz$.

   Hint: Campbell’s theorem can be useful: Let $\Phi$ be a homogeneous Poisson process in $\mathbb{R}^d$ with intensity $\lambda$ and $f : \mathbb{R}^d \to \mathbb{R}$ a non-negative measurable function. Then it holds that
   $$\mathbb{E} \sum_{v \in \Phi} f(v) = \lambda \int_{\mathbb{R}^d} f(z) \, dz.$$

   Further, it holds that
   $$\mathbb{E} \sum_{v, w \in \Phi \atop v \neq w} f(v) g(w) = \lambda^2 \int_{\mathbb{R}^d} f(z) \, dz \int_{\mathbb{R}^d} g(z) \, dz.$$

3. Consider the Ising model given in the lecture.
   a) Prove that the Ising model has the Markov property with respect to the 4-neighbourhood relation $\sim_4$. Does the Ising model possess the Markov property with respect to the 8-neighbourhood relation $\sim_8$?
b) For any $\Gamma \subset T_N$ the joint distribution of the system of random variables $\{X_t, t \in \Gamma\}$ is denoted by $P^\Gamma_{T_N}$, i.e. for $\Gamma = \{t_1, \ldots, t_n\}$ and $x_{t_1}, \ldots, x_{t_n} = \pm 1$ it holds that $P^\Gamma_{T_N}(x_{t_1}, \ldots, x_{t_n}) = P(X_{t_1} = x_{t_1}, \ldots, X_{t_n} = x_{t_n})$. For any such $\Gamma$ the expectation $\sigma_\Gamma = E \prod_{t \in \Gamma} X_t$ is called correlation function.

Prove that the probabilities $P^\Gamma_{T_N}$ can be expressed in terms of the correlation functions $\sigma_\Gamma$ as follows:

$$P^\Gamma_{T_N}(x_{t_1}, \ldots, x_{t_n}) = (-1)^k 2^n \sum_{\Gamma' \subset \Gamma} C_{\Gamma'} \sigma_{\Gamma'}$$

where $k$ is the number of values $x_{t_i}$ equal to $-1$ and $C_{\Gamma'} = \prod_{t \in \Gamma \setminus \Gamma'} x_t$.

c) Assume that $J = 0$, i.e. there is no interaction between the particles. Prove that for any $\Gamma \subset T_N$ the correlation function $\sigma_\Gamma$ is given by

$$\sigma_\Gamma = \left( \frac{\exp\left(-\frac{mB}{K_0\alpha}\right) - \exp\left(\frac{mB}{K_0\alpha}\right)}{\exp\left(-\frac{mB}{K_0\alpha}\right) + \exp\left(\frac{mB}{K_0\alpha}\right)} \right)^{\vert \Gamma \vert}.$$ 

4. Prove that the covariance function $C(t, s) = E X_t X_s - E X_t E X_s$ of a random field $X$ is positive semi-definite, i.e. $\forall n \in \mathbb{N}, \forall t_1, \ldots, t_n \in T, \forall z_1, \ldots, z_n \in \mathbb{R}$ it holds that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} C(t_i, t_j) z_i z_j \geq 0.$$ 

5. The process $X = \{X_t, t \in [0, \infty)\}$ is defined as follows: $X_0$ takes values $+1$ and $-1$ with probability $1/2$ each and $X_t = X_0 \cdot (-1)^Y([0,t])$ for $t > 0$ where $Y$ is a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda > 0$, independent of $X_0$. See Figure 1 (a) for a trajectory of $X$.

a) Compute the expectation function of $X$ and its covariance function.

b) Let’s assume that the process $X$ takes not only values $\pm 1$ but when the $k$th point of $Y$ occurs, it assumes a random value $V_k$ and retains that value till the next point of $Y$ occurs, i.e. $X_0 = V_0$ and $X_t = V_Y([0,t])$ for $t > 0$ where $V_0, V_1, \ldots$ are i.i.d. random variables with $V_0 \sim \mathcal{N}(0, \sigma^2)$. Figure 1 (b) shows a trajectory of $X$. Specify the marginal distribution. Is it a Gaussian process?
(a) Trajectory of the process $X$ in Exercise 5 a).

(b) Trajectory of the process $\tilde{X}$ in Exercise 5 b).

Figure 1: Illustration for Exercise 5.