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Homework assignment #4 for Random Fields I

Due Thursday, June 18, 2009

1. *One-dimensional random walk:*
   a) Let Y denote a homogeneous Poisson process on $[0, \infty)$ with intensity λ. Let $X_1, X_2, \ldots$ i.i.d. random variables, independent of Y with $\mathbb{E}X_1 = 0$ and $\operatorname{Var} X_1 = \sigma^2$. Define the random process Z on $[0, \infty)$ by $Z_t = \sum_{i=1}^{Y_t} X_i$. Compute the expectation function and the covariance function of Z.
   b) Consider the process $\widetilde{Z}_t = Y_t - \lambda t$. Compare the expectation, the covariance and the trajectories of Z and $\widetilde{Z}$.

   Hint: Wald's identity could be useful: Let N be a random variable with $P(N \in \mathbb{N}) = 1$ and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables independent of N. Assume that $\mathbb{E}N$ and $\mathbb{E}X_0$ exist. Then $\mathbb{E}\sum_{n=1}^{N} X_n$ exists and is given by $\mathbb{E}\sum_{n=1}^{N} X_n = \mathbb{E}N\mathbb{E}X_0$. Further, if $\operatorname{Var}(X_1)$ and $\operatorname{Var}(N)$ exist, it holds that $\operatorname{Var}\left(\sum_{n=1}^{N} X_n\right) = \operatorname{Var}(X_1)\,\mathbb{E}N + (\mathbb{E}X_1)^2 \operatorname{Var}(N)$.

2. a) Compute all cumulants of a $\mathcal{N}(\mu, \sigma^2)$-distributed random variable.
   b) Compute all cumulants of a Poisson-distributed random variable with parameter λ.
   c) Prove that the cumulants $\langle X^n \rangle$ and the moments μ_n of a random variable X with $\mathbb{E}|X|^n < \infty$ are related by the following recursive formula:

$$\langle X^n \rangle = \mu_n - \sum_{i=1}^{n-1} \binom{n-1}{i-1} \langle X^i \rangle \mu_{n-i}$$

3. Prove the following properties of cumulants:
   a) Symmetry: For any permutation $\sigma : (1, \ldots, n) \mapsto (\sigma(1), \ldots, \sigma(n))$ it holds that

$$S_X(t_1, \ldots, t_n, r) = S_X\left(t_{\sigma(1)}, \ldots, t_{\sigma(n)}, r_\sigma\right).$$

   b) Multilinearity: For all $a, b \in \mathbb{R}$ it holds that

$$\langle aX_{t_0} + bX_{t_1}, X_{t_2}, \ldots, X_{t_n} \rangle = a \langle X_{t_0}, X_{t_2}, \ldots, X_{t_n} \rangle + b \langle X_{t_1}, X_{t_2}, \ldots, X_{t_n} \rangle.$$

c) If $\{t_1, \ldots, t_n\} = A \cup B$, $A \cap B = \emptyset$, $A, B \neq \emptyset$ such that $\{X_t, t \in A\}$ and $\{X_t, t \in B\}$ are independent, then $S_X(t_1, \ldots, t_n, r) = 0$ for all $r \in \mathbb{N}^n$.

d) It holds that $C(s,t) = \langle X_s, X_t \rangle$ for $s, t \in T$.

4. Give an example for a random field which is stationary in the wide sense but not strictly stationary.

5. Consider the Shot-Noise process $X_t = \sum_{v \in \Phi} g(t - v)$.

   a) Prove that X_t is stationary in the wide sense.

   b) Find conditions on the function g such that the process X_t is weakly isotropic.