Homework assignment #4 for Random Fields I

Due Thursday, June 18, 2009

1. One-dimensional random walk:
   a) Let $Y$ denote a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda$. Let $X_1, X_2, \ldots$ i.i.d. random variables, independent of $Y$ with $\mathbb{E} X_1 = 0$ and $\text{Var} X_1 = \sigma^2$. Define the random process $Z$ on $[0, \infty)$ by $Z_t = \sum_{i=1}^{Y_t} X_i$.

   Compute the expectation function and the covariance function of $Z$.

   b) Consider the process $\tilde{Z}_t = Y_t - \lambda t$. Compare the expectation, the covariance and the trajectories of $Z$ and $\tilde{Z}$.

   **Hint:** Wald’s identity could be useful: Let $N$ be a random variable with $\mathbb{P}(N \in \mathbb{N}) = 1$ and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables independent of $N$. Assume that $\mathbb{E} N$ and $\mathbb{E} X_0$ exist. Then $\mathbb{E} \sum_{n=1}^{N} X_n$ exists and is given by $\mathbb{E} \sum_{n=1}^{N} X_n = \mathbb{E} N \mathbb{E} X_0$. Further, if $\text{Var} (X_1)$ and $\text{Var} (N)$ exist, it holds that $\text{Var} \left( \sum_{n=1}^{N} X_n \right) = \text{Var} (X_1) \mathbb{E} N + (\mathbb{E} X_1)^2 \text{Var} (N)$.

2. a) Compute all cumulants of a $\mathcal{N}(\mu, \sigma^2)$-distributed random variable.

   b) Compute all cumulants of a Poisson-distributed random variable with parameter $\lambda$.

   c) Prove that the cumulants $\langle X^n \rangle$ and the moments $\mu_n$ of a random variable $X$ with $\mathbb{E} |X|^n < \infty$ are related by the following recursive formula:

   $$\langle X^n \rangle = \mu_n - \sum_{i=1}^{n-1} \binom{n-1}{i-1} \langle X^i \rangle \mu_{n-i}$$

3. Prove the following properties of cumulants:

   a) Symmetry: For any permutation $\sigma : (1, \ldots, n) \mapsto (\sigma (1), \ldots, \sigma (n))$ it holds that

   $$S_X (t_1, \ldots, t_n, r) = S_X (t_{\sigma (1)}, \ldots, t_{\sigma (n)}, r_{\sigma})$$

   b) Multilinearity: For all $a, b \in \mathbb{R}$ it holds that

   $$\langle aX_{t_0} + bX_{t_1}, X_{t_2}, \ldots, X_{t_n} \rangle = a \langle X_{t_0}, X_{t_2}, \ldots, X_{t_n} \rangle + b \langle X_{t_1}, X_{t_2}, \ldots, X_{t_n} \rangle.$$
c) If \{t_1, \ldots, t_n\} = A \cup B, A \cap B = \emptyset, A, B \neq \emptyset\ such\ that\ \{X_t, t \in A\} \ and \ \{X_t, t \in B\} \ are\ independent,\ then\ S_X (t_1, \ldots, t_n, r) = 0\ for\ all\ r \in \mathbb{N}^n.

d) It holds that \( C(s, t) = \langle X_s, X_t \rangle \) for \( s, t \in T \).

4. Give an example for a random field which is stationary in the wide sense but not strictly stationary.

5. Consider the Shot-Noise process \( X_t = \sum_{v \in \Phi} g(t - v). \)

   a) Prove that \( X_t \) is stationary in the wide sense.

   b) Find conditions on the function \( g \) such that the process \( X_t \) is weakly isotropic.