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Homework assignment #5 for Random Fields I

Due Thursday, June 25, 2009

1. The *cosine field* X on $\mathbb{R}^d$ can be defined by

$$X(t) = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \left(Y_k \cos(a_k t_k) + Z_k \sin(a_k t_k) \right)$$

   with i.i.d. centered random variables $Y_1, \ldots, Y_d, Z_1, \ldots, Z_d$ and positive constants $a_1, \ldots, a_d$. Prove that the cosine field X is weakly stationary.

2. Prove that a homogeneous Poisson process on the real line is stochastically continuous but it does not possess an almost surely continuous modification (i.e. all its realizations are discontinuous).

3. Let $X = \{X_t, t \in \mathbb{R}^+\}$ be a process with the following properties:
   - $X_0 = 0$ almost surely
   - X has independent increments
   - there is $\sigma^2 > 0$ such that $X_t - X_s \sim \mathcal{N}(0, \sigma^2 |t - s|)$, $\forall t, s > 0$.

   This process is called *Wiener process*. For $\sigma^2 = 1$ it is called *standard Wiener process*. Prove that the Wiener process is almost surely continuous.

4. Let X be a stochastically continuous random field on a compact index space T, i.e. X is stochastically continuous for all $t \in T$. Prove that X is *uniformly stochastically continuous on* T, i.e. for all $\varepsilon, \eta > 0$ $\exists\, \delta > 0 : \forall s, t \in T$ with $|s - t|_T < \delta$ it holds that $P(|X_s - X_t|_E > \varepsilon) \leq \eta$.

5. Construct a stationary shot-noise process with almost surely discontinuous realizations which is continuous in quadratic mean.