Prof. Dr. E. Spodarev
D. Meschenmoser

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Homework assignment #6 for Random Fields I

Due Thursday, July 02, 2009

1. Give an example of a random process which is not stochastically continuous.
2. Prove Lemma 1.5.2 given in the lecture: For any $t_0 \in T$ and some random variable Y with $\mathbb{E}Y^2 < \infty$, the following statements are equivalent:
   1) $X_s \xrightarrow{L^2} Y$ as $s \to t_0$
   2) $C(s,t) \longrightarrow \mathbb{E}Y^2$ as $s,t \to t_0$

   Hint: For 1) $\Rightarrow$ 2)*, first prove the following fact: If* $X_n \xrightarrow{L^2} X$ *and* $Y_n \xrightarrow{L^2} Y$ *for some random sequences* $\{X_n\}$ *and* $\{Y_n\}$ *and random variables* X *and* Y*, then* $\mathbb{E}(X_m Y_n) \to \mathbb{E}(XY)$ *as* $n, m \to \infty$*. For* 2) $\Rightarrow$ 1)*, prove that* $\{X_s\}$ *is a fundamental sequence in the* L^2*-sense as* $s \to t_0$*.*
3. Show that
   a) the Wiener process is not even stochastically differentiable at any $t \in [0, \infty)$.
   b) the homogeneous Poisson process on the real line is stochastically differentiable, but not in p-mean $\forall p > 1$.
4. Let $T = \mathbb{N}_0$ and E be a countable phase space. The process $X = \{X_t, t \in T\}$ is called a *(discrete-time) Markov chain* if it satisfies the *Markov property*: for any $n \geq 1$, any $t, t_1, \ldots, t_n \in T$ with $t_1 < \ldots < t_n < t$, and any $i_1, \ldots, i_n, j \in E$,
   $$\mathbb{P}(X_t = j \,|X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = \mathbb{P}(X_t = j \,|X_{t_n} = i_n).$$
   The *initial distribution* $\alpha = (\alpha_j, j \in E)$ is given by $\alpha_j = \mathbb{P}(X_0 = j)$ and the *transition probabilities* $p_{ij}(s,t)$ are given by $p_{ij}(s,t) = \mathbb{P}(X_t = j \,|X_s = i)$ for $t \geq s$, and $i, j \in E$. The chain X is called *homogeneous*, if $p_{ij}(s,t)$ depends on s and t only through $t - s$. Then it is enough to know α and the 1-step *transition matrix* $P = (p_{ij})$ with $p_{ij} = \mathbb{P}(X_{n+1} = j \,|X_n = i)$, $n \geq 0$. The *n-step transition matrix* $P^{(n)}$ is given by $P^{(n)} = \left(p_{ij}^{(n)}\right)$ with the *n-step transition probabilites* $p_{ij}^{(n)} = \mathbb{P}(X_n = j \,|X_0 = i)$. The *Chapman-Kolmogorov equation* states that $P^{(n+m)} = P^{(n)} P^{(m)}$ for any $n, m \in \mathbb{N}$.
   a) Prove the Chapman-Kolmogorov equation.
   b) Give an example for a stochastic process with $T = \mathbb{N}_0$ and countable phase space E which satisfies the Chapman-Kolmogorov equation but does not fulfil the Markov property.