Homework assignment #6 for Random Fields I

Due Thursday, July 02, 2009

1. Give an example of a random process which is not stochastically continuous.

2. Prove Lemma 1.5.2 given in the lecture: For any \( t_0 \in T \) and some random variable \( Y \) with \( \mathbb{E} Y^2 < \infty \), the following statements are equivalent:
   
   1) \( X_s \xrightarrow{L^2} Y \) as \( s \to t_0 \)
   
   2) \( C(s,t) \xrightarrow{} \mathbb{E} Y^2 \) as \( s,t \to t_0 \)

   Hint: For 1) \( \Rightarrow \) 2), first prove the following fact: If \( X_n \xrightarrow{L^2} X \) and \( Y_n \xrightarrow{L^2} Y \) for some random sequences \( \{X_n\} \) and \( \{Y_n\} \) and random variables \( X \) and \( Y \), then \( \mathbb{E}(X_nY_n) \to \mathbb{E}(XY) \) as \( n,m \to \infty \). For 2) \( \Rightarrow \) 1), prove that \( \{X_s\} \) is a fundamental sequence in the \( L^2 \)-sense as \( s \to t_0 \).

3. Show that
   
   a) the Wiener process is not even stochastically differentiable at any \( t \in [0, \infty) \).
   
   b) the homogeneous Poisson process on the real line is stochastically differentiable, but not in \( p \)-mean \( \forall p > 1 \).

4. Let \( T = \mathbb{N}_0 \) and \( E \) be a countable phase space. The process \( X = \{X_t, t \in T\} \) is called a (discrete-time) Markov chain if it satisfies the Markov property: for any \( n \geq 1 \), any \( t, t_1, \ldots, t_n \in T \) with \( t_1 < \ldots < t_n < t \), and any \( i_1, \ldots, i_n, j \in E \),
   
   \[ \mathbb{P}(X_t = j \mid X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = \mathbb{P}(X_t = j \mid X_{t_n} = i_n) . \]

   The initial distribution \( \alpha = (\alpha_j, j \in E) \) is given by \( \alpha_j = \mathbb{P}(X_0 = j) \) and the transition probabilities \( p_{ij}(s,t) \) are given by \( p_{ij}(s,t) = \mathbb{P}(X_t = j \mid X_s = i) \) for \( t \geq s \), and \( i, j \in E \). The chain \( X \) is called homogeneous, if \( p_{ij}(s,t) \) depends on \( s \) and \( t \) only through \( t - s \). Then it is enough to know \( \alpha \) and the 1-step transition matrix \( P = (p_{ij}) \) with \( p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) \), \( n \geq 0 \). The n-step transition matrix \( P^{(n)} \) is given by \( P^{(n)} = \left(p_{ij}^{(n)}\right) \) with the n-step transition probabilites \( p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) \). The Chapman-Kolmogorov equation states that \( P^{(n+m)} = P^{(n)}P^{(m)} \) for any \( n, m \in \mathbb{N} \).
   
   a) Prove the Chapman-Kolmogorov equation.
   
   b) Give an example for a stochastic process with \( T = \mathbb{N}_0 \) and countable phase space \( E \) which satisfies the Chapman-Kolmogorov equation but does not fulfil the Markov property.