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Homework assignment #7 for Random Fields I

Due Thursday, July 09, 2009

1. A function $f : \mathbb{R}^d \to \mathbb{C}$ is called *positive definite* if

$$\sum_{k=1}^{n} \sum_{j=1}^{n} a_k f\left(x_k - x_j\right) \overline{a_j} \geq 0$$

for any $x_k \in \mathbb{R}^d$, $a_k \in \mathbb{C}$ and for all $n \in \mathbb{N}$. Here, $\overline{a}$ denotes the complex conjugate of a.

Prove that the characteristic function of any random variable is a valid covariance function, i.e. it is positive definite.

2. Let $C_1, C_2, \ldots$ be positive definite functions and let $a_1, a_2 > 0$.
   a) *Linear combination:* Prove that $a_1 C_1 + a_2 C_2$ is positive definite.
   b) *Multiplication:* Prove that $C_1 C_2$ is positive definite.
   c) *Pointwise limit:* Let $C(t) = \lim_{n\to\infty} C_n(t)$ for all $t \in \mathbb{R}^d$. Prove that C is a covariance function.

3. Prove *Pólya's criterion:* If $C : [0, \infty) \to \mathbb{R}$ is such that $C(0) = 1$, $C(t)$ is continuous and convex, and $\lim_{t\to\infty} C(t) = 0$, then C is a positive definite function.

4. Are the following functions $C : \mathbb{R} \to \mathbb{R}$ valid covariance functions?
   a) $C(t) \equiv 0$
   b) $C(t) \equiv 1$
   c) *Nugget effect:* $C(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$
   d) $C(t) = \begin{cases} 1, & t = 0 \\ 1/2, & t \neq 0 \end{cases}$
   e) $C(t) = \sin(t)$
   f) $C(t) = \begin{cases} \exp(-|t|), & t \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$