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Homework assignment #8 for Random Fields I

Due Thursday, July 16, 2009

1. Prove the following statement: A function $C : \mathbb{R} \to \mathbb{C}$ is real, continuous, and positive definite, if and only if it is the cosine transform of a non-negative finite symmetric measure F on $[0, \infty)$, i.e.

$$C(x) = \int_{[0,\infty)} \cos(xt)\, dF(t) \qquad \text{for all } x \in \mathbb{R}.$$

2. If the measure F in Bochner's theorem has a density f, it holds that

$$C(x) = \int_{\mathbb{R}} f(t) \exp(itx)\, dt,$$

i.e. C equals 2π times the inverse Fourier transform of f. The function f is called *spectral density* and is given by

$$f(t) = \int_{\mathbb{R}} C(x) \exp(-ixt)\, dx.$$

Compute the spectral density f for the following covariance functions $C : \mathbb{R} \to \mathbb{R}$:

a) $C(x) = \exp(-x^2)$

b) $C(x) = \exp(-|x|)$

c) $C(x) = \begin{cases} 1 - \frac{|x|}{2} & , -2 \le x \le 2 \\ 0 & , \text{otherwise} \end{cases}$

3. Prove that the following functions are valid covariance functions.

a) $C(x) = \cos(x)$ on $\mathbb{R}$

b) *Resacling:* $C(x) = C_1(ax)$ where C_1 is a positive definite function on $\mathbb{R}^d$ and $a \in \mathbb{R}$

c) *Scale mixture:* $C(x) = \int_{[0,\infty)} C_1(ax)\, dF(a)$ for a non-negative measure F and a covariance function C_1