Corollary 11.5. If $N \sim \widetilde{NB}(\gamma, p)$ and $X_1 \sim Bin(1, r)$, then $S := X_1 + \ldots + X_N \sim \widetilde{NB}\left(\gamma, \frac{p}{p+(1-p)r}\right)$.

Proof. $X_1 \sim Bin(1,r) \Rightarrow P[X_1 = 1] = r,\ P[X_1 = 0] = 1-r$

$$\Rightarrow \varphi_{X_1}(t) = \mathbb{E}e^{tX_1} = r \cdot e^{it} + (1-r)e^{0\cdot t} = re^{it} + (1-r).$$

Cor. 11.4 implies that

$$\varphi_S(t) = g_N(\varphi_{X_1}(t)) = \left(\frac{p}{1-(1-p)(1-r+re^{it})}\right)^{\gamma}. \quad (*)$$

Now, the characteristic function of a $\widetilde{NB}\left(\gamma, \frac{p}{p+(1-p)r}\right)$-distr. ~~is given by~~ can be computed as follows:

Let $L \sim \widetilde{NB}(\gamma, q)$, where $q = \frac{p}{p+(1-p)r}$. Then,

$$g_L(t) = \left(\frac{q}{1-(1-q)t}\right)^{\gamma} \quad \text{(see the proof of Cor. 11.4).}$$

It follows that the char. func. of L is given by

$$\varphi_L(t) = \mathbb{E}e^{iLt} = \mathbb{E}(e^{it})^L = g_L(e^{it}) = \left(\frac{q}{1-(1-q)e^{it}}\right)^{\gamma}. \quad (**)$$

Hence,

$$\varphi_L(t) \overset{(**)}{=} \left(\frac{\frac{p}{p+(1-p)r}}{1-\left(1-\frac{p}{p+(1-p)r}\right)e^{it}}\right)^{\gamma} = \left(\frac{p}{p+(1-p)r-(1-p)re^{it}}\right)^{\gamma}$$

$$= \left(\frac{p}{1-(1-p)(1-r)-(1-p)re^{it}}\right)^{\gamma} = \left(\frac{p}{1-(1-p)(1-r+re^{it})}\right)^{\gamma}$$

$$\overset{(*)}{=} \varphi_S(t). \text{ Hence, } S \overset{\text{distr.}}{=} L. \qquad \square$$