Junior-Prof. Dr. Z. Kabluchko
Wolfgang Karcher

Summer term 2010
5th April 2010

Risk Theory

Exercise Sheet 3

Due to: 12th May 2010

Exercise 1 (6 points) *Distribution function of the Erlang distribution*

Let $X \sim \text{Gamma}(k, \lambda)$ be a Gamma–distributed random variable, where $\lambda > 0$ and $k > 0$ is *integer*. Show that the distribution function of X is given by

$$F_X(t) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \quad t \geq 0.$$

Remark. The $\text{Gamma}(k, \lambda)$ distribution with $k \in \mathbb{N}$ is the distribution of the arrival time of the k-th claim in the Poisson point process with intensity λ. It is also called the Erlang distribution.

Exercise 2 (6 points) *Expected number of claims in a mixed Poisson process*

Let $(N(t))_{t \geq 0}$ be a mixed Poisson process with mixing parameter Λ, where Λ is a random variable taking positive values with probability 1.

(a) Give a formula for $\mathbb{P}[N(t) = k]$ where $t > 0$, $k \in \mathbb{N} \cup \{0\}$.

(b) Assume that $\mathbb{E}\Lambda < \infty$. Show that $\mathbb{E}[N(t)] = (\mathbb{E}\Lambda)t$.

Exercise 3 (6 points) *Bernoulli process as a claim arrival process*

Let $W_1, W_2, \ldots$, be independent random variables having the geometric distribution with parameter $p \in (0, 1]$, i.e., $\mathbb{P}[W_i = n] = p(1-p)^{n-1}$, $n \in \mathbb{N}$, $i \in \mathbb{N}$. Let $T_n = W_1 + \ldots + W_n$, $n \in \mathbb{N}$, and define a stochastic process $(X_i)_{i \in \mathbb{N}}$ by

$$X_i = \begin{cases} 1, & \text{if } T_n = i \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let also $Y_1, Y_2, \ldots$ be a sequence of independent random variables with $\mathbb{P}[Y_i = 1] = p$ and $\mathbb{P}[Y_i = 0] = 1 - p$, $i \in \mathbb{N}$. Show that the stochastic processes $(X_i)_{i \in \mathbb{N}}$ and $(Y_i)_{i \in \mathbb{N}}$ have the same finite-dimensional distributions.

Hint: Given $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, compute the probabilities $\mathbb{P}[X_1 = \varepsilon_1, \ldots, X_n = \varepsilon_n]$ and $\mathbb{P}[Y_1 = \varepsilon_1, \ldots, Y_n = \varepsilon_n]$.

Exercise 4 (6 points) *Poisson point process and the Beta distribution*

Let $T_0, T_1, T_2, \ldots$ be a Poisson point process with intensity λ. For $k, n \in \mathbb{N}$ with $k \leq n$, show that $T_k/T_n \sim \text{Beta}(k, n-k)$, i.e., the density of T_k/T_n is given by

$$f_{T_k/T_n}(t) = \frac{(n-1)!}{(k-1)!(n-k-1)!} t^{k-1}(1-t)^{n-k-1}, \quad t \in [0, 1].$$

Hint: Use without proof the following formula: If $Y, Z > 0$ are random variables with joint density $f_{Y,Z}$, then the density of Y/Z is given by $f_{Y/Z}(t) = \int_0^\infty s f_{Y,Z}(st, s) ds$.