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Risk Theory

Exercise Sheet 4

Due to: 19th May 2010

Exercise 1 (6 points) *A characterization of the Bin, Poi, and $\widetilde{NB}$ distributions*

Given a random variable X with values in $\mathbb{N}\cup\{0\}$, we define $q_n = \mathbb{P}[X = n]$, $n \in \mathbb{N}\cup\{0\}$. Prove that if $X \sim \text{Bin}(n, p)$ for some $n \in \mathbb{N}$, $p \in (0, 1)$ or $X \sim \widetilde{NB}(c, p)$ for some $c > 0$, $p \in (0, 1)$, then there exist $a, b \in \mathbb{R}$ such that $q_n = (a + \frac{b}{n})q_{n-1}$ for all $n \in \mathbb{N}$.

Remark. Recall that we write $X \sim \widetilde{NB}(c, p)$ if $\mathbb{P}[X = n] = \binom{n+c-1}{n}p^c(1-p)^n$ for all $n \in \mathbb{N}\cup\{0\}$.

Exercise 2 (6 points) *Recursion equations for geometric and logarithmic distributions*

(a) A random variable X has logarithmic distribution with parameter $p \in (0, 1)$ (notation: $X \sim \text{Log}(p)$), if

$$\mathbb{P}[X = n] = -\frac{1}{\log(1-p)} \cdot \frac{p^n}{n}, \quad n \in \mathbb{N}.$$

Show that this is indeed a well-defined probability distribution and compute its expectation.

(b) Given a random variable X with values in $\mathbb{N}$, we define $q_n = \mathbb{P}[X = n]$, $n \in \mathbb{N}$. Show that if $X \sim \text{Geo}(p)$ or $X \sim \text{Log}(p)$, then there exist $a, b \in \mathbb{R}$ such that $q_n = (a + \frac{b}{n})q_{n-1}$ for all $n = 2, 3, \ldots$.

Exercise 3 (6 points) *First claim arrival in a Beta-mixed Bernoulli process*

Consider the following experiment: first, a random variable $P \sim \text{Beta}(\alpha, \beta)$ is generated. Then, a coin having the probability P of landing heads is tossed infinitely often. Let T_1 be the time at which it lands heads for the first time. Show that

$$\mathbb{P}[T_1 = n] = \frac{B(\alpha + 1, \beta + n - 1)}{B(\alpha, \beta)}, \quad n \in \mathbb{N}.$$

Remark. A random variable X is said to have Beta distribution with parameters $\alpha, \beta > 0$ (notation: $X \sim \text{Beta}(\alpha, \beta)$), if the density of X is given by

$$f_X(t) = \frac{1}{B(\alpha, \beta)} t^{\alpha-1}(1-t)^{\beta-1} 1_{t\in(0,1)}.$$

Here, $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ is the Beta function.

Exercise 4 (6 points) *Negative hypergeometric distribution*

(a) Consider an urn containing R red and $N - R$ green balls. Fix some number $r \in \mathbb{N}$, $1 \leq r \leq R$. We draw the balls from the urn without replacement (ohne Zurücklegen) until the number of red balls we have collected reaches the value r. Let X be the total number of balls we have drawn. Compute the probability $\mathbb{P}[X = n]$ for all $n = r, r+1, \ldots$.

(b) The distribution of X is called negative hypergeometric. What detail in the description of the experiment has to be changed in order to obtain the negative binomial (resp. hypergeometric) distribution?