Risk Theory
Exercise Sheet 7
Due to: 9th June 2010

Exercise 1 (6 points) Max-stable distributions

(a) Let $X_1, X_2, \ldots$ be independent random variables with Gumbel distribution, that is, $P[X_i \leq t] = e^{-e^{-t}}$ for every $t \in \mathbb{R}$. Show that for every $n \in \mathbb{N}$, the random variable $Z_n := \max(X_1, \ldots, X_n) - \log n$ has Gumbel distribution as well.

(b) Let $X_1, X_2, \ldots$ be independent random variables having a Fréchet distribution with parameter $\alpha$, that is, $P[X \leq t] = e^{-1/t^\alpha}$ for every $t > 0$. Show that the random variable $Z_n := n^{-1/\alpha} \max(X_1, \ldots, X_n)$ has Fréchet distribution with parameter $\alpha$ as well.

Exercise 2 (6 points) Benktander type I distribution

A random variable $X$ has Benktander type I distribution with parameters $a, b > 0$, where $a(a+1) \geq 2b$, if its tail function is given by

$$F_X(t) = a^{-1}t^{-a-1}e^{-b \log^2 t}(a + 2b \log t), \quad t \geq 1.$$  

Compute the mean excess function of $X$.

Remark: The Log-normal distribution has no closed-form mean excess function. The Benktander type I distribution provides a model having the same type of tail behavior as the Log-normal distribution and a simple mean excess function.

Exercise 3 (6 points) Expectation for insurance with retention

Let $X$ be a risk with mean excess loss function $e_X$.

(a) Show that $E\max(X - d, 0) = (1 - P(X > d))e_X(d)$ for $d > 0$.

(b) Show that $E(X) = (1 - P(X > d))e_X(d) + E(\min(X, d))$.

Remark. Consider an insured risk $X$ with retention level (Selbstbehalt) $d$. Then, $Y := \max(X - d, 0)$ is the part that an insurance company has to pay, while $Z := \min(X, d)$ is the part that the insurance holder has to pay.

Hint. You may use without proof the formula $EZ = \int_0^\infty F_Z(t)\,dt$ valid for any non-negative random variable $Z$.

Exercise 4 (6 points) The Poisson point process paradox

Let $T_0, T_1, T_2, \ldots$ be a Poisson point process. Let $s > 0$ and $N(s) = \sum_{t \leq s} 1_{T_t \leq s}$. Compute the distribution function $F_{Z_s}$ of $Z_s := T_{N(s)+1} - T_{N(s)}$ as well as the limit $\lim_{s \to +\infty} F_{Z_s}(t)$ for $t > 0$. 