The application of probabilistic method in graph theory
Probabilistic method

The **probabilistic method** is a nonconstructive method, primarily used in combinatorics and pioneered by Paul Erdős, for proving the existence of a prescribed kind of mathematical object.

There are two ways to use the probabilistic method:

1. Showing the probability of the mathematical object is greater than 0
2. Calculating the exception of some random variable
Was ist der Graph?

Graph is an ordered pair \( G = G(V, E) \), comprising a set \( V \) of vertices together with a set \( E \) of edges.

If the graph is directed, \( E \) is a subset of Cartesian product \( V \times V \).

If the graph is undirected, \( E \) is a two-element subset of \( V \) or empty set \( \emptyset \).

Complete Graph \( G(V,E) \) (directed):

*For all* \( x, y \in V, x \neq y \): \( (x,y) \in E \) and \( (x,x) \) doesn't exist for all \( x \in V \)

Subgraph \( G^*(V^*,E^*) \) of graph \( G(V,E) \):

\[
V^* \subseteq V \text{ and } E^* \subseteq E
\]
The application of probabilistic method in graph theory | Jiayi Li | 10.06.2010

undirected graph

directed graph
Ramsey number $R(k,l)$

Def.

The smallest integer $n$ such that in any 2-coloring (here red and blue) of the edges of a complete graph on $n$ vertices, there either is a red $K_k$ (a complete subgraph on $k$ vertices, all of whose edges are colored red), or there is a blue $K_l$. 

$R(3,3) = 6$

Vertex = 5

Vertex = 6
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Proposition

If \( \binom{n}{k} 2^{1-\binom{k}{2}} < 1 \), then \( R(k, k) > n \). Thus \( R(k,k) > 2^{\frac{k}{2}} \) for all \( k \geq 3 \).
Proof

For any fixed set $R$ of $k$ vertices, let $A_R$ be the event that the induced subgraph of $K_n$ on $R$ is **monochromatic** (either all its edges are red or blue). Then the probability of $P_R$ is

$$\Pr(A_R) = 2^{1 - \binom{k}{2}}$$

If

$$\binom{n}{k} 2^{1 - \binom{k}{2}} < 1$$

thus with positive probability, no events $A_R$ occurs and there is a 2-coloring of $K_n$, that is, $R(k,k) > n$. 
Note that if $k \geq 3$ and we take $n = \left\lfloor \frac{k}{2^2} \right\rfloor$, then

$$\binom{n}{k} \cdot 2^{1 - \binom{k}{2}} < 2^{1 + \frac{k}{2}} \cdot \frac{n^k}{k!} \cdot \frac{k^2}{2^{21}} < 1$$

and hence $R(k, k) > 2^2$ for all $k \geq 3$. 
Linearity of Expectation

Let $X_1, \ldots, X_n$ be random variables, $X = c_1 X_1 + \ldots + c_n X_n$. Linearity of expectation states that

$$E[X] = c_1 E[X_1] + \cdots + c_n E[X_n]$$

Let $\sigma$ be a random permutation on $\{1, \ldots, n\}$, uniformly chosen. Let $X$ be the number of fixed points of $\sigma$. Then

$$E[X_i] = \Pr[\sigma(i) = i] = \frac{1}{n},$$

where $X_i$ is the indicator random variable of the event $\sigma(i) = i$. 
We decompose $X = X_1 + \ldots + X_n$, so that

$$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{1}{n} + \ldots + \frac{1}{n} = 1.$$
Tournament

A **tournament** is a graph obtained by assigning a direction for each edge in an undirected complete graph.
Hamiltonian path = a path that visits each vertex exactly once
$\sigma = (1,3,5,2,4)$

Hamiltonian path  
not Hamiltonian path
Theorem

There is a tournament $T$ (graph) with $n$ players (vertices) and at least $n! \cdot 2^{-(n-1)}$ Hamiltonian path.
Proof

Let $X$ be the number of Hamiltonian paths. For each permutation $\sigma$, let $X_\sigma$ be the indicator random variable for $\sigma$ giving a Hamiltonian path. Then $X = \sum X_\sigma$ and

$$E[X] = \sum_{\sigma} E[X_\sigma] = n!2^{-\binom{n}{2}}$$

Thus some tournament has at least $E[X]$ Hamiltonian paths.
**Def.**

\[
girth(G) = \text{the size of the smallest circuit in graph } G
\]

\[
\alpha(G) = \text{the size of the largest independent set in } G
\]

\[
\chi(G) = \text{the smallest number of colors needed to color a Graph } G
\]
girth(G) = 6

α(G) = 4, χ(G) = 3
Theorem (Erdös [1959])

For all $k, l$ there exists a graph $G$ with $\text{girth}(G) > l$ and $\chi(G) > k$. 
Proof

Fix \( \theta < 1/l \), let \( G \sim G(n, p) \) with probability \( p = n^{\theta-1} \). Let \( X \) be the number of circuits of size at most \( l \). Then

\[
E[X] = \sum_{i=3}^{l} \frac{n!}{2i \cdot (n-i)!} p^i \leq \sum_{i=3}^{l} \frac{n^{\theta i}}{2i}
\]

With Markov’s inequality we know

\[
\Pr\left[X \geq \frac{n}{2}\right] \leq \frac{2}{n} E[X] \rightarrow 0, \text{if } n \rightarrow \infty
\]
Set \( x = \left(\frac{3}{p}\right) \ln n \) so that

\[
\Pr[\alpha(G) \geq x] \leq \binom{n}{x} (1 - p)^{\binom{x}{2}} < \left[ e^{\ln n - p(x-1)/2} \right]^x \rightarrow 0, \text{if } n \to \infty
\]

If \( n \) is large enough, the probability of these two events is less than. Then there exists a \( G \) with less than \( n/2 \) cycles of length less than \( l \) and with \( \alpha(G) < 3n^{1-\theta} \ln n \).

Now we should use a trick, from \( G \) a vertex from each cycle of length at most \( l \) to remove. The new graph \( G^* \) have at least \( n/2 \) vertices and its girth is greater than \( l \). Thus

\[
\chi(G^*) \geq \frac{|G^*|}{\alpha(G^*)} \geq \frac{n/2}{3n^{1-\theta} \ln n} = \frac{n^\theta}{6 \ln n}
\]

If \( N \) large enough, then \( \chi(G^*) > k \)
Conclusion

Although the proof uses probability, the final conclusion is determined for certain, without any possible error.
Vielen Dank für Ihre Aufmerksamkeit!