# The application of probabilistic method in graph theory


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# Probabilistic method

The **probabilistic method** is a nonconstructive method, primarily used in combinatorics and pioneered by Paul Erdős, for proving the existence of a prescribed kind of mathematical object.

There are two ways to use the probabilistic method：

1. Showing the probability of the mathematical object is greater than 0

2. Calculating the exception of some random variable

## Was ist der Graph?

Graph is an ordered pair G = G(V , E) , comprising a set V of vertices together with a set E of edges.

If the graph is directed , E is a subset of Cartesian product V x V.

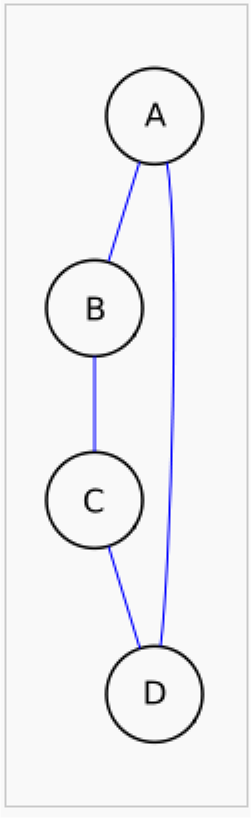
If the graph is undirected , E is a two-element subset of V or empty set Ø.

Complete Graph G(V,E) (directed):

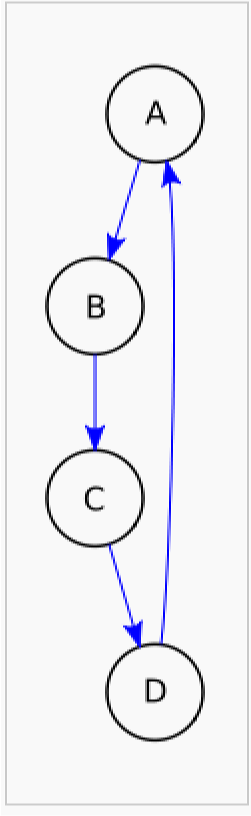
$$For\ all\ x, y \in V, \mathrm{x} \neq \mathrm{y}\text{: (x,y)} \in E \text{ and (x,x) doesn't exisit for all x} \in V$$

Subgraph G*(V*,E*) of graph G(V,E):

$$V^* \subseteq V \text{ and } \mathrm{E}^* \subseteq \mathrm{E}$$




undirected graph



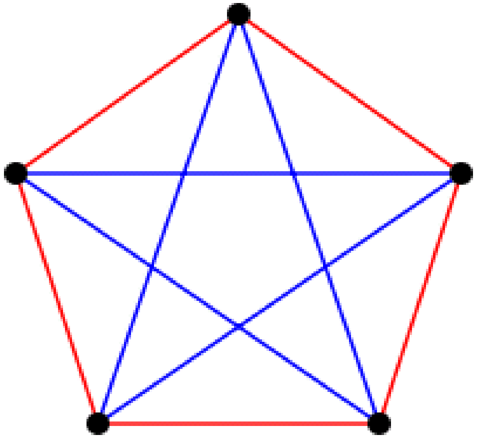

directed graph

# Ramsey number R(k,l)

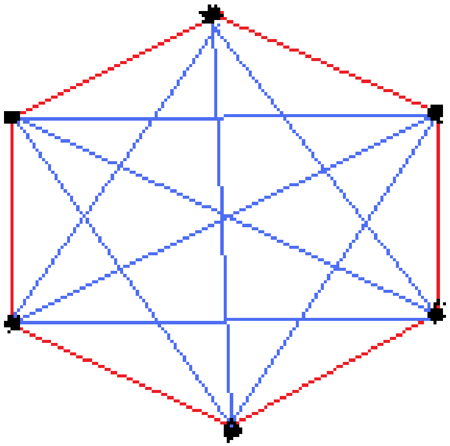
Def.

The smallest integer n such that in any 2-coloring (here red and blue) of the edges of a complete graph on n vertices, there either is a red $K_k$ (a complete subgraph on k vertices , all of whose edges are colored red), or there is a blue $K_l$ .

## R(3,3) = 6



Vertex = 5



Vertex = 6

| $r, s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **2** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| **3** | 1 | 3 | 6 | 9 | 14 | 18 | 23 | 28 | 36 | 40 − 43 |
| **4** | 1 | 4 | 9 | 18 | 25 | 35 − 41 | 49 − 61 | 56 − 84 | 73 − 115 | 92 − 149 |
| **5** | 1 | 5 | 14 | 25 | 43 − 49 | 58 − 87 | 80 − 143 | 101 − 216 | 125 − 316 | 143 − 442 |
| **6** | 1 | 6 | 18 | 35 − 41 | 58 − 87 | 102 − 165 | 113 − 298 | 127 − 495 | 169 − 780 | 179 − 1171 |
| **7** | 1 | 7 | 23 | 49 − 61 | 80 − 143 | 113 − 298 | 205 − 540 | 216 − 1031 | 233 − 1713 | 289 − 2826 |
| **8** | 1 | 8 | 28 | 56 − 84 | 101 − 216 | 127 − 495 | 216 − 1031 | 282 − 1870 | 317 − 3583 | 317-6090 |
| **9** | 1 | 9 | 36 | 73 − 115 | 125 − 316 | 169 − 780 | 233 − 1713 | 317 − 3583 | 565 − 6588 | 580 − 12677 |
| **10** | 1 | 10 | 40 − 43 | 92 − 149 | 143 − 442 | 179 − 1171 | 289 − 2826 | 317-6090 | 580 − 12677 | 798 − 23556 |

# Proposition

$$If\ \binom{n}{k} 2^{1-\binom{k}{2}} < 1,\ then\ R(k,k) > n.\ Thus\ \mathrm{R(k,k)} > 2^{\frac{k}{2}}\ \text{for all k} \geq 3.$$

## Proof

For any fixed set R of k vertices, let $A_R$ be the event that the induced subgraph of $K_n$ on R is *monochromatic* (either all its edges are red or blue) .Then the probability of Pr is

$$\Pr(A_R) = 2^{1-\binom{k}{2}}$$

If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ ,thus with positive probability, no events $A_R$ occurs and there is a 2-coloring of $K_n$, that is, R(k,k) > n.

Note that if $k \geq 3$ and we take $n = \left\lfloor 2^{\frac{k}{2}} \right\rfloor$ ,then

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < \frac{2^{1+\frac{k}{2}}}{k!} \cdot \frac{n^k}{2^{\frac{k^2}{2}}} < 1$$

and hence $R(k,k) > 2^{\frac{k}{2}}$ for all $k \geq 3$.

## Linearity of Expectation

Let $X_1,\ldots,X_n$ be random variables , $X = c_1X_1+\ldots+c_nX_n$. Linearity of expectation states that

$$E[X]=c_1E[X_1]+\cdots+c_nE[X_n]$$

Let σ be a random permutation on{1,…,n}, uniformly chosen. Let X be the number of fixed points of σ. Then

$$E[X_i]=\Pr[\sigma(i)=i]=\frac{1}{n} \quad ,$$

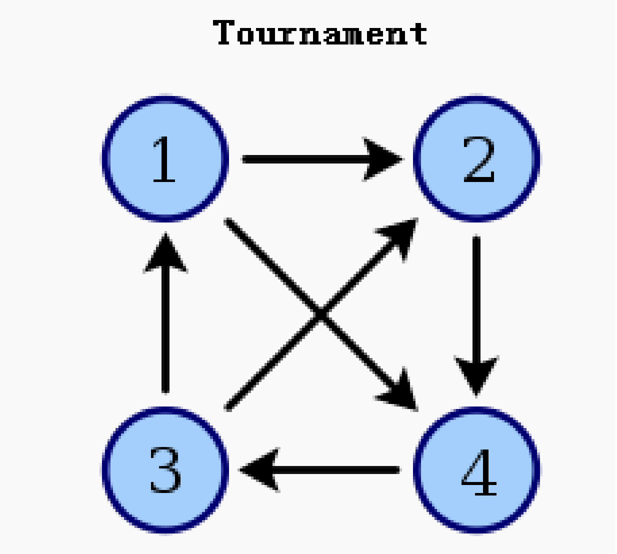
where Xi is the indicator random variable of the event σ(i)=i.

We decompose $X = X_1 + \ldots + X_n$ , so that

$$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{1}{n} + \ldots + \frac{1}{n} = 1.$$

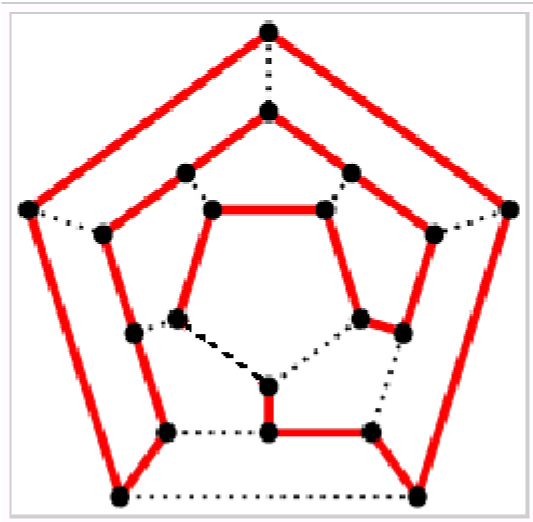
# Tournament

A **tournament** is a graph obtained by assigning a direction for each edge in an undirected complete graph.




A tournament on 4 vertices

Hamiltonian path = a path that visits each vertex exactly once



# $\sigma = (1,3,5,2,4)$

$1 \longrightarrow 3 \longrightarrow 5 \longrightarrow 2 \longrightarrow 4$

Hamiltonian path

$1 \longrightarrow 3 \longleftarrow 5 \longleftarrow 2 \longrightarrow 4$

not Hamiltonian path

# Theorem

There is a tournament T (graph) with n players (vertices) and at least $n!2^{-(n-1)}$ Hamiltonian path.

# Proof

Let X be the number of Hamiltonian paths . For each permutation σ, let $X_\sigma$ be the indicator random variable for σ giving a Hamiltonian path. Then $X=\Sigma X_\sigma$ and

$$E[X]=\sum_{\sigma}E[X_\sigma]=n!2^{-(n-1)}$$

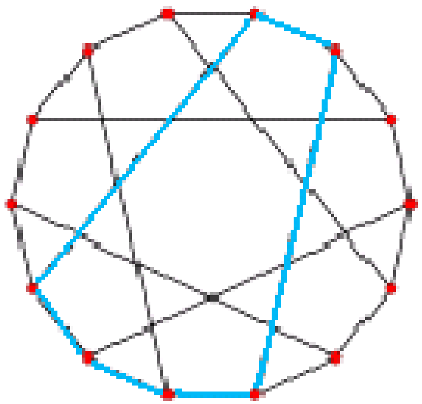
Thus some tournament has at least E[X] Hamiltonian paths.

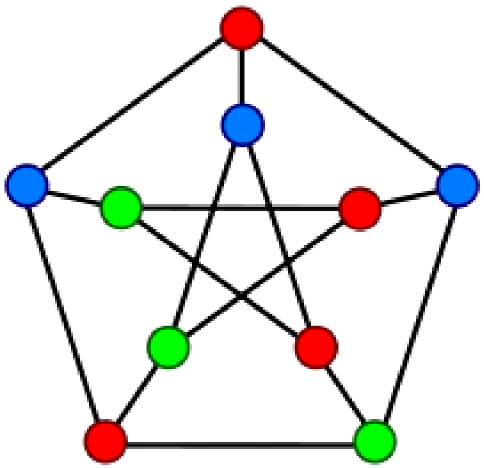
# Def.

$\mathrm{girth}(G) =$ the size of the smallest circuit in graph G

$\alpha(G) =$ the size of the largest independent set in G

$\chi(G) =$ the smallest number of colors needed to color a Graph G





girth(G) = 6

α(G) = 4 ,χ(G) = 3

# Theorem (Erdös [1959])

For all $k$ , $l$ there exists a graph $G$ with $\mathrm{girth}(G) > l$ and $\chi(G) > k$.

# Proof

Fix θ < 1/l , let G ~ G (n , p) with probability p = $n^{\theta-1}$. Let X be the number of circuits of size at most l. Then

$$E[X] = \sum_{i=3}^{l} \frac{n!}{2i \cdot (n-i)!} p^{i} \le \sum_{i=3}^{l} \frac{n^{\theta i}}{2i}$$

With Markov's inequality we know

$$\Pr\left[ X \ge {}^{n}\!/_{2} \right] \le \frac{2}{n} E[X] \to 0, if \;\; n \to \infty$$

$Set\ \mathrm{x} = \lceil (3/p)\ln n \rceil$ so that

$$\Pr\left[\alpha(G) \geq x\right] \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left[e^{\ln n - p(x-1)/2}\right]^{x} \to 0, if\ \ n \to \infty$$

If n is large enough , the probability of these two events is less than . Then there exists a G with less than n/2 cycles of length less than l and with $\alpha(G) < 3n^{1-\theta}$ lnn.

Now we should use a trick ,from G a vertex from each cycle of length at most l to remove. The new graph **G*** have at least n/2 vertices and its girth is **greater than l**. Thus

$$\chi(G^*) \geq \frac{|G^*|}{\alpha(G^*)} \geq \frac{n/2}{3n^{1-\theta}\ln n} = \frac{n^{\theta}}{6\ln n}$$

If N large enough ,then **χ(G*) > k**

# Conclusion

**Although the proof uses probability, the final conclusion is determined for *certain*, without any possible error.**

# Vielen Dank für Ihre Aufmerksamkeit!