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Zufällige Graphen

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Contents

Contents

The Probabilistic Method

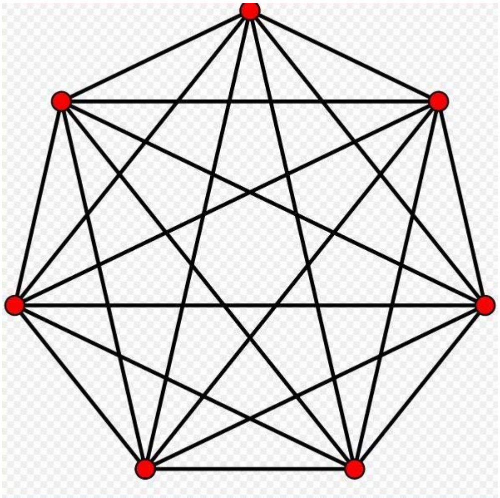
Trying to prove that a structure with certain desired properties exists, one defines an appropriate probility space of structures and then shows that the desired properties hold in this space with positive probability.

Content

Complete Graph

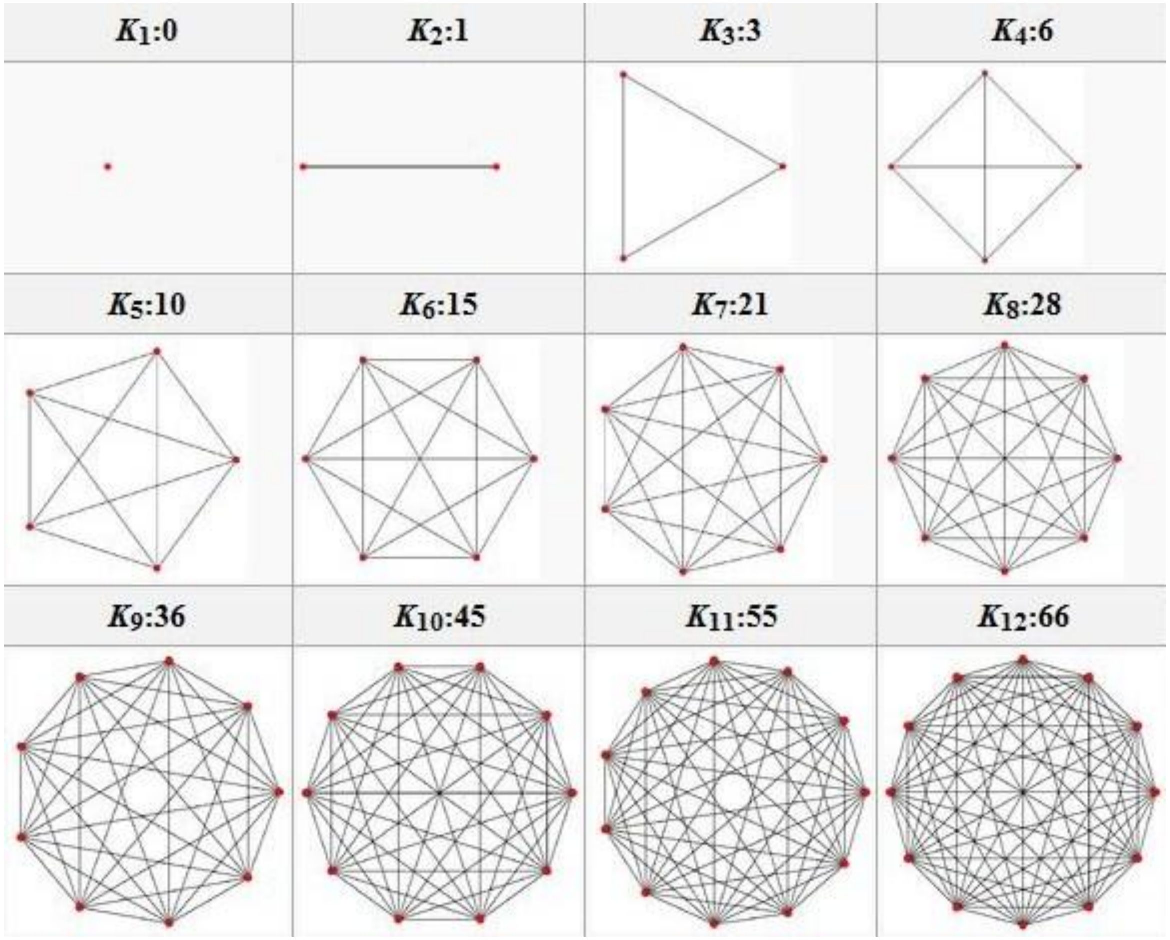
A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge.

The complete graph on n vertices has $n(n-1)/2$ edges, and is denoted by K_n



Examples

Complete graphs on *n* vertices, for *n* between 1 and 12, are shown below along with the numbers of edges:



Subgraph

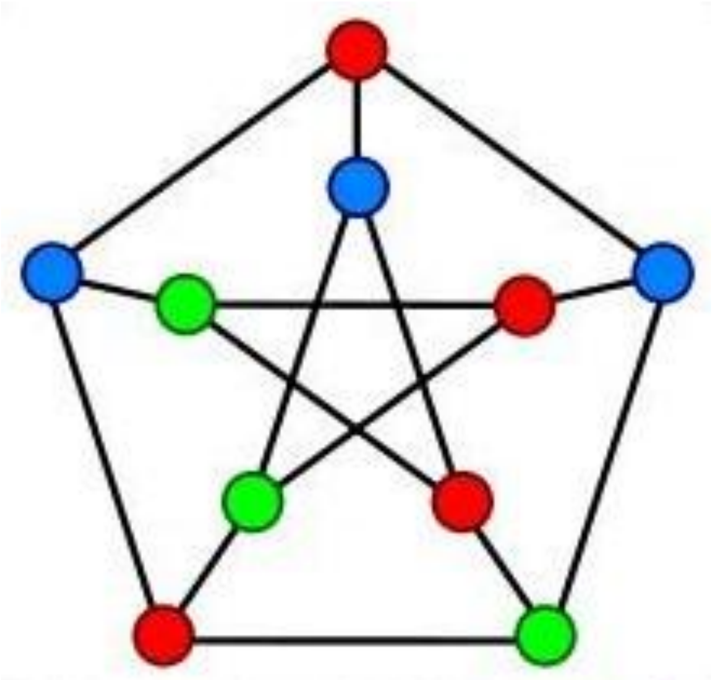
A subgraph of a graph G is a graph whose vertex set is a subset of that of G, and whose adjacency relation is a subset of that of G restricted to this subset.

Graph coloring

Graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints.

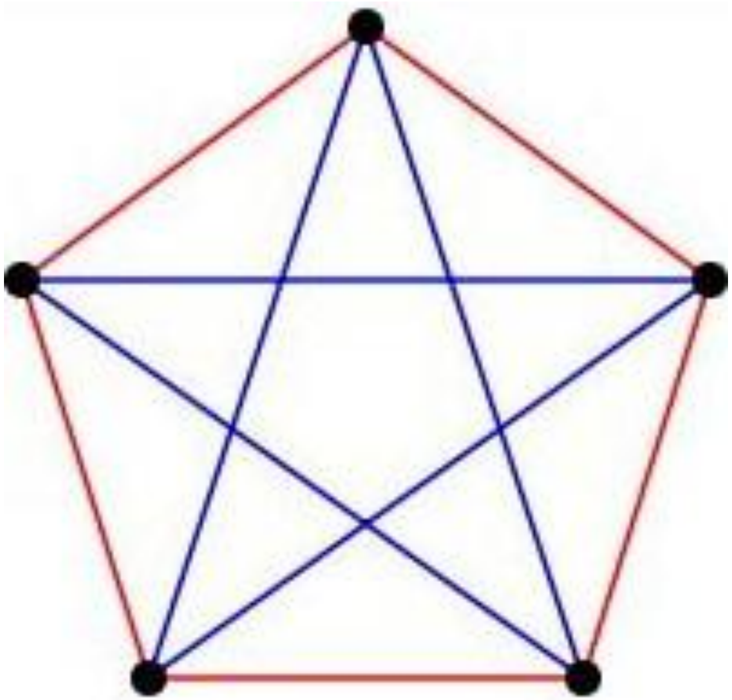
Graph coloring

In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges share the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.



A proper vertex coloring of the Petersen graph with 3 colors, the minimum number possible.

Example: 2-coloring of the edges of a complete graph



The Ramsey Number $R(k,l)$

The *Ramsey number* $R(k,l)$ is the smallest integer *n* such that in any 2-coloring of the edges of a complete graph on *n* vertices K_n by red and blue, there either is a red K_k (i.e., a complete subgraph on *k* verticies, all of whose edges are colored red), or there is a blue K_l.

Ramsey(1930) showed that $R(k,l)$ is finite for any two integers *k* and *l.*

A lower bound for the diagonal Ramsey numbers $R(k,k)$

If $\binom{n}{k}\cdot 2^{1-\binom{k}{2}}<1$, then $R(k,k)>n$. Thus $R(k,k)>2^{k/2}$ for all $k\geq 3$.

A lower bound for the diagonal Ramsey numbers $R(k,k)$

Proof:

Consider a random 2-coloring of the edges of K_n ;

For any fixed set *R* of *k* vertices ;

Let A_R be the event : the induced subgraph of K_n on *R* is *monochromatic*

Clearly, $\mathrm{P}(A_R) = 2^{1-\binom{k}{2}}$

There are $\binom{n}{k}$ possible choices for *R*

$\Longrightarrow$ P(at least one of the events A_R occurs)) $\leq \binom{n}{k} 2^{1-\binom{k}{2}} < 1$

$\Longrightarrow$ P(no event A_R occurs) > 0

$\Longrightarrow$ There is a 2-coloring of the edges of K_n without a *monochromatic* K_k

$\Longrightarrow$ $R(k,k) > n$

A lower bound for the diagonal Ramsey numbers $R(k,k)$

Proof:

Note that $k \geq 3$ and we take $n = \lfloor 2^{k/2} \rfloor$, then

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{2^{1+k/2}}{k!} \cdot \frac{n^k}{2^{k^2/2}} < 1,$$

and hence $R(k,k) > 2^{k/2}$ for all $k \geq 3$. ■

This simple example demonstrates the essence of the probabilistic method. To prove the existence of a good coloring we do not present one explicitly, but rather show, in a nonconstructive way, that it exists.

“Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens. ”

—Joel Spencer

Contents

Basic

Let $X_1, \ldots, X_n$ be random variables, $X = c_1 X_1 + \cdots + c_n X_n$.

Linearity of expectation states that :

$$E[X] = c_1 E[X_1] + \cdots + c_n E[X_n]$$

The power of this principle comes from there being no restrictions on the dependence or independence of the X_i .

Permutation

In algebra and particularly in group theory, a permutation of a set *S* is defined as a bijection from *S* to itself (i.e., a map $S \rightarrow S$ for which every element of *S* occurs exactly once as image value). To such a map *f* is associated the rearrangement of *S* in which each element *s* takes the place of its image *f*(*s*).

Thus there are six permutations of the set {1,2,3}, namely [1,2,3], [1,3,2], [2,1,3], [2,3,1], [3,1,2], and [3,2,1].

Example

Let σ be a random permutation on {1,…,n}, uniformly chosen.

Let X be the number of fixed points of σ.

To find EX , we decompose $X = X_1 + \cdots + X_n$, where X_i is the indicator random variable of the event $\sigma(i) = i$. Then

$$EX_i = \mathrm{P}\left[\sigma(i) = i\right] = \frac{1}{n}$$

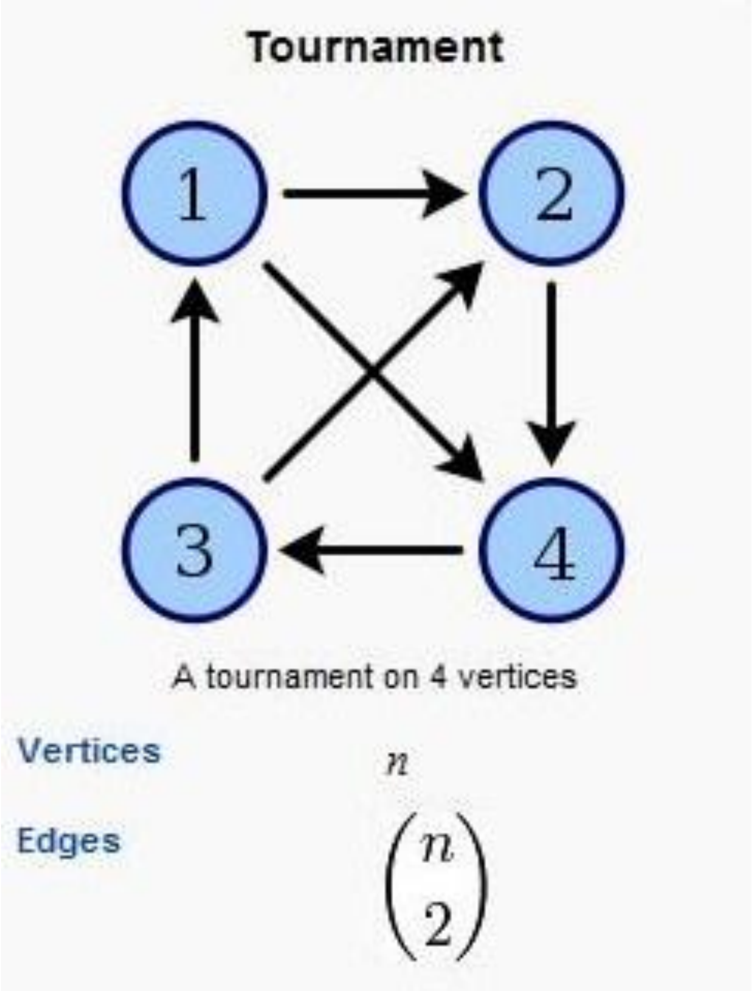
so that

$$EX = \frac{1}{n} + \cdots + \frac{1}{n} = 1$$

In application we often use the fact that there is an event in the probability space for which $X \geq EX$ and an event for which $X \leq EX$.

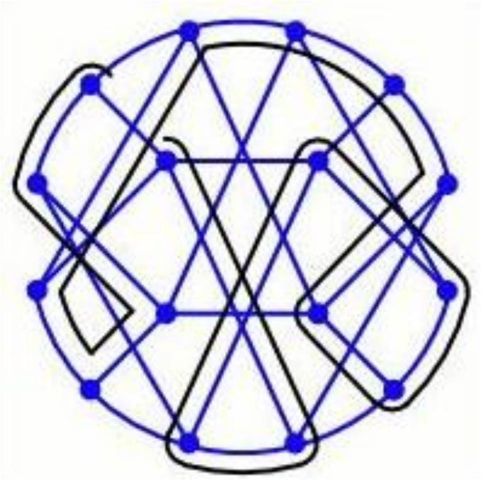
Tournament

A tournament is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is a directed graph in which every pair of vertices is connected by a single directed edge.



Hamiltonian Paths

A Hamiltonian path is a path in a graph which visits each vertex exactly once.



A Hamiltonian path (black) over a graph (blue).

Hamiltonian Path in Tournament T

Theorem: *There is a tournament T with n players and at least* $n!2^{-(n-1)}$ *Hamiltonian paths.*

Proof.

In the random tournament, let *X* be the number of Hamiltonian paths.

For each permutation σ, let X_σ be the indicator random variable for σ giving a Hamiltonian path.

—that is, satisfying $(\sigma(i), \sigma(i+1)) \in T$ for $1 \le i < n$.

Then $X = \sum X_\sigma$ and

$$EX = \sum EX_\sigma = n!2^{-(n-1)}.$$

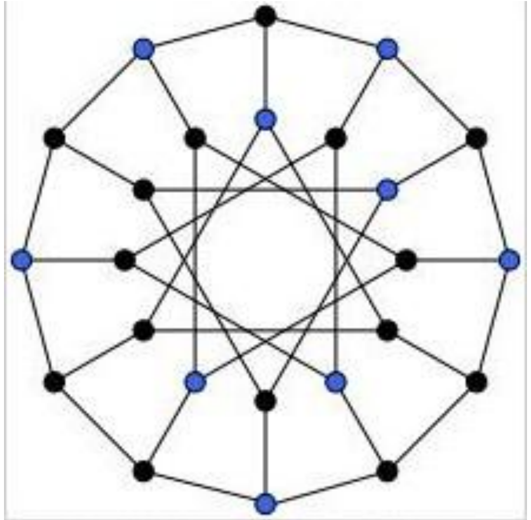
Thus some tournament has at least EX Hamiltonian paths. ■

Contents

Independent set

In graph theory, an independent set is a set of vertices in a graph, no two of which are adjacent. That is, it is a set *I* of vertices such that for every two vertices in *I*, there is no edge connecting the two. Equivalently, each edge in the graph has at most one endpoint in *I*. The size of an independent set is the number of vertices it contains.

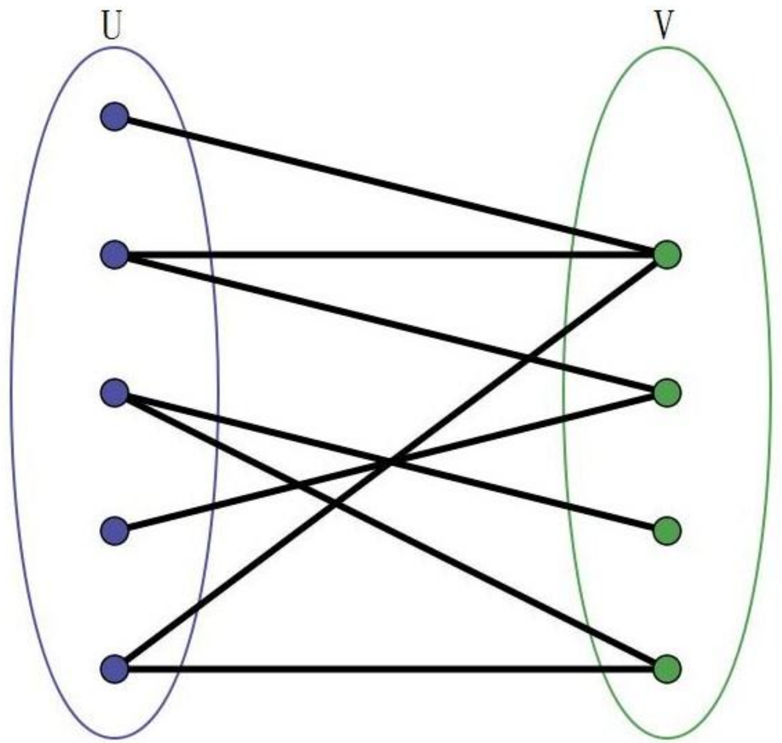
A **maximum independent set** is a largest independent set for a given graph *G* and its size is denoted ***α*(*G*)**.



The nine blue vertices form a maximum independent set for the Generalized Petersen graph GP(12,4).

Bipartite Graph

A bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets.



Theorem

Theorem : *Let* $G=(V,E)$ *be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least e/2 edges.*

Proof.

Let $T \subseteq V$ be a random subset such that $\mathrm{P}\left[x \in T\right]=1/2$;

Set $B=V-T$;

Call an edge {*x*,*y*} crossing if exactly one of *x*,*y* are in *T* ;

Let *X* be the number of crossing edges ;

We decompose

$$X = \sum_{\{x,y\}\in E} X_{xy},$$

Where X_{xy} is the indicater random variable for {*x*,*y*} being crossing .

Theorem

Theorem : *Let* $G=(V,E)$ *be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least e/2 edges.*

Proof.

Then

$$EX_{xy}=\frac{1}{2}$$

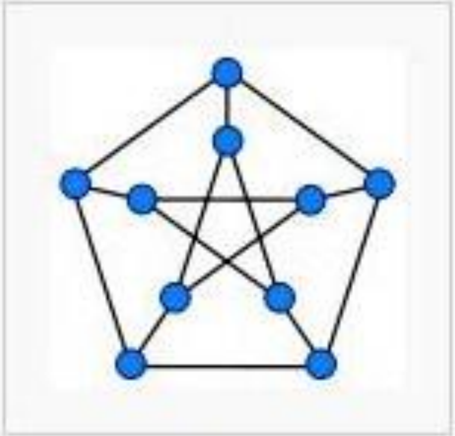
$$\Rightarrow \qquad EX=\sum_{\{x,y\}\in E} EX_{xy}=\frac{e}{2}$$

Thus $X\geq e/2$ for some choice of *T*, and the set of those crossing edges form a bipartite graph. ■

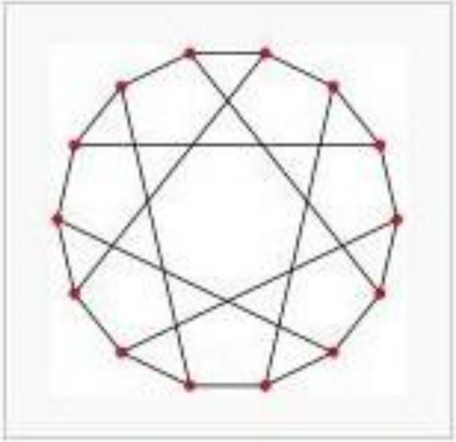
Contents

Girth

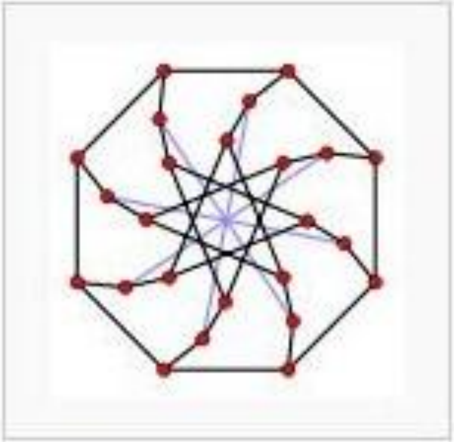
In graph theory, the girth of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles, its girth is defined to be infinity. For example, a 4-cycle (square) has girth 4. A grid has girth 4 as well, and a triangular mesh has girth 3. A graph with girth >3 is triangle-free.



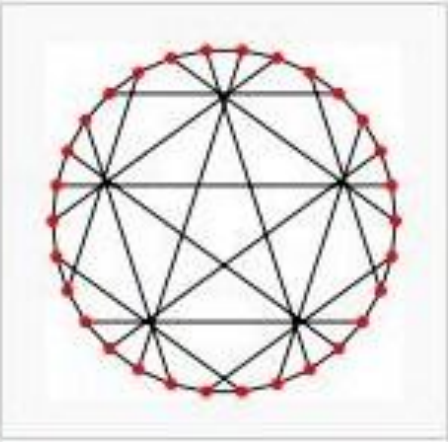
The Petersen graph has a girth of 5



The Heawood graph has a girth of 6



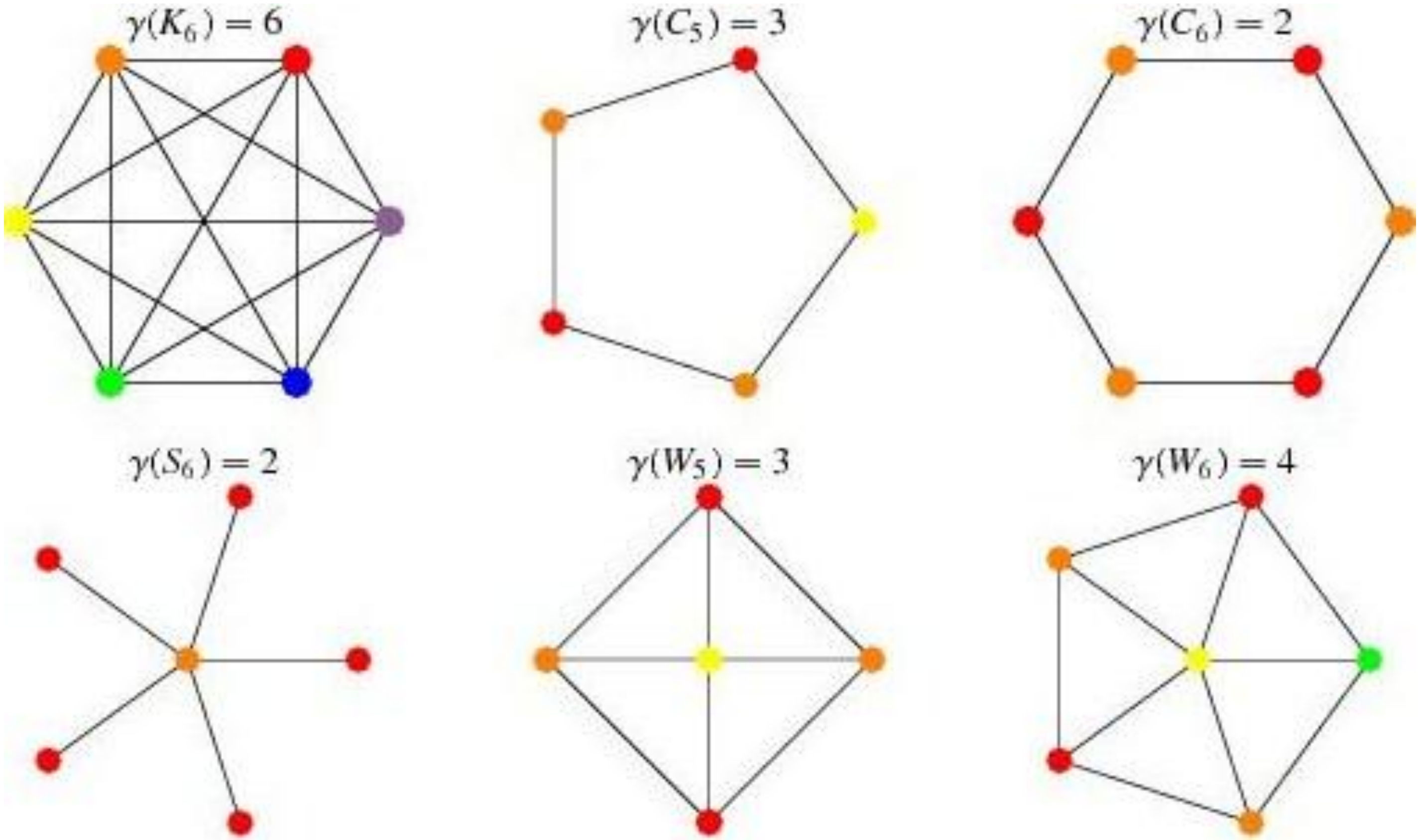
The McGee graph has a girth of 7



The Tutte eight cage has a girth of 8

Chromatic Number

The chromatic number of a graph *G* is the smallest number of colors $\gamma(G)$ needed to color the vertices of *G* so that no two adjacent vertices share the same color, i.e., the smallest value of *k* possible to obtain a *k*-coloring.



Theorem (Erdős [1959])

Theorem : *For all k,l there exists a graph G with girth(G) > l and* $\gamma(G) > k$*.*

Proof.

Fix $\theta < 1/l$;

Let $G \sim G(n, p)$ with $p = n^{\theta - 1}$ (That is, *G* is a random graph on *n* vertices chosen by picking each pair of vertices as an edge randomly and independently with probability *p*) ;

Let *X* be the number of circuits of size at most *l*. Then

$$EX = \sum_{i=3}^{l} \frac{n!}{(n-i)! \cdot 2i} p^i \leq \sum_{i=3}^{l} \frac{n^{\theta i}}{2i} = o(n)$$

as $\theta l < 1$.

Theorem (Erdős [1959])

For all k,l there exists a graph G with girth(G) > l and $\gamma(G) > k$.

Proof.

In particular,

$$\mathrm{P}\left[X \geq n/2\right] = o(1)$$

Set $x = \lceil (3/p)\ln n \rceil$ so that

$$\mathrm{P}\left[\alpha(G) \geq x\right] \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left[ne^{-p(x-1)/2}\right]^{x} = o(1).$$

Let *n* be sufficiently large so that both these events have probability less than 1/2.

Theorem (Erdős [1959])

For all k,l there exists a graph G with girth(G) > l and $\gamma(G) > k$.

Proof.

Then there is a specific *G* with less than *n/2* cycles of length less than *l* and with $\alpha(G) < 3n^{1-\theta} \ln n$.

Remove from *G* a vertex from each cycle of length at most *l*. This gives a graph G^* with at least *n/2* vertices. G^* has girth greater than *l* and $\alpha(G^*) \leq \alpha(G)$. Thus

$$\gamma(G^*) \geq \frac{|G^*|}{\alpha(G^*)} \geq \frac{n/2}{3n^{1-\theta} \ln n} = \frac{n^\theta}{6 \ln n}.$$

To complete the proof, let *n* be sufficiently large so that this is greater than *k*. ■

Vielen Dank für Ihre Aufmerksamkeit!