Historical Root of Stochastic Geometry

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Introduction

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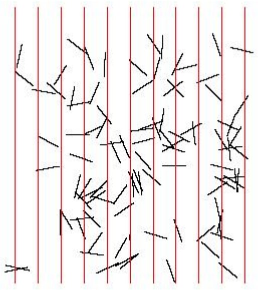
Buffon's needle

Suppose we have a floor made of parallel wooden strips, each the same width and that we drop a needle onto the floor.

What is the probability that the needle will lie across a line between two strips?



Buffon's needle

Solution:

L: the length of the needle
d: the distance from the center of the needle to the nearest line
: the width of the strip, and θ: the angle that the needle makes with the direction of the parallel lines

(i)$L \leq D$
The probability space is $R = [0, D/2] \times [0, \pi]$. Find the lebesgue measure of an appropriate subset of R and divide it by $\lambda(R)$.
$\lambda(R)$: Lebesgue measure of R.
The needle crosses a line if $d \leq L \sin\theta/2$, so the probability of the crossing is given by

$$\frac{\int_0^{\pi} L/2 \ \sin\theta \, d\theta}{\lambda(R)} = 2L/(\pi D).$$

Buffon's needle

(ii) $L > D$ Omitted.

In the Buffon's needle problem, we assumed that all sample objects were equally likely, however, such a definition requires careful consideration.

Buffon's needle

We can solve the Buffon's problem by using Expectation.

$\mathbb{E}(x) = kx$ for some constant k, where $\mathbb{E}(x)$ is the expected number of crossings for a segment of length x.

Next, we consider a circle with diameter D. A circle of a diameter of D will always cross a line in two places. $k = 2/(\pi D)$. Thus $\mathbb{E}(L) = 2L/(\pi D)$. In this case, the expected value is just the probability of intersecting a line.

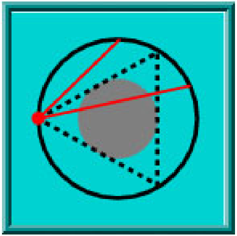
Bertrand's paradox

Bertrand's paradox

Suppose we have an equilateral triangle inscribed in a circle and that a chord of the circle is chosen randomly. What is the probability that the chord is longer than a side of the length?

Bertrand's paradox

1. The random endpoint method
Choose randomly two points on a circle and measure the distance between the two. One point can be chosen anywhere on the circle without loss of generality.



Bertrand's paradox

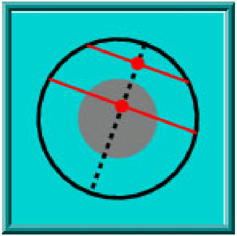
Consider an equilateral triangle with three sides of equal length $\sqrt{3}$ whose vertex is on the diameter. Call the vertices A, B and C. Draw a chord from A. If the other point of the chord is between B and C, the length of the chord is longer than $\sqrt{3}$. The probability is $1/3$.

Bertrand's paradox

Bertrand's paradox

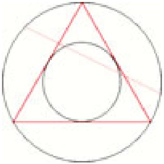
2. The random midpoint method (II)

Given a radius and a side of a triangle perpendicular to the radius. Assume a uniform distribution over the midpoint positions of parallel chords perpendicular to the radius. By symmetry, it is clear that chords whose length is more than $\sqrt{3}$ have their midpoints nearer the center of the circle than half the radius. The probability is $1/2$.



Bertrand's paradox

3. Assume a uniform distribution over the midpoint positions of chords, which are not necessary parallel each other. The chord is longer than a side of an equilateral triangle if the chosen point is within the smaller circle on the figure below. First, fix the direction. Consider only chords parallel each other. Then, change the direction continuously up to 360° The probability is 1/4. Therefore, Bertrand's paradox tells answers change depending on methods of random selection.



Integral geometry

Consider a straight line G in the plane.

$$x \cos \phi + y \sin \phi - p = 0, \tag{1}$$

p: the distance to the origin
ϕ: the direction to the closest point
$0 \leq p$ and $0 \leq \phi < 2\pi$.
Seek a measure on a set of lines that is invariant under a rotation followed by a translation.

The measure of a set of lines $G(p, \phi)$ is given by $dG = dpd\phi$.

It is easy to check the measure is invariant under a rotation followed by a translation.

Rigid motions of the Euclidean Plane

Suppose we have (x',y') after giving to (x, y) a motion given by a rotation α followed by a translation by the vector (x_0, y_0).

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, therefore$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' - x_0 \\ y' - y_0 \end{pmatrix}$$

Substituting the above formula into (1), we have
$p + \cos(\phi+\alpha)x_0 + \sin(\phi+\alpha)y_0 = \cos(\phi+\alpha)x' + \sin(\phi+\alpha)y'$.

Rigid motions of the Euclidean Plane

Thus a new coordinate (p', ϕ') can be written in the following way.
$p' = p + \cos(\phi + \alpha)x_0 + \sin(\phi + \alpha)y_0$
$\phi' = \phi + \alpha$

since the original line equation was given by
$x \cos \ \phi + y \ \sin \ \phi = p$.
We rotated the line α anticlockwise, and the distance between the original point and the line changed to p'.

Jacobian formula

The Jacobian formula for the change in measure is given by

$dp'd\phi' = |J|dpd\phi$, where

$$|J| = \left|\frac{\partial(p',\phi')}{\partial(p,\phi)}\right| = \begin{vmatrix} \frac{\partial p'}{\partial p} & \frac{\partial p'}{\partial \phi} \\ \frac{\partial \phi'}{\partial p} & \frac{\partial \phi'}{\partial \phi} \end{vmatrix} = 1$$

Hence, we have shown that the measure is invariant under a rotation followed by a translation.

Measure of the set of lines

Let D be a domain in the plane of area F and let $G(p, \phi)$ be the measure of a set of lines. dG is called the density for sets of lines.

Multiplying both side of $dG = dpd\phi$ by the length σ of the chord $G \cap D$ and integrating over all the lines G, we have

$$\int_{G \cap D \neq \emptyset} \sigma \, dG = \pi F.$$

Lines that intersect a convex set

Theorem

The measure of the set of lines that intersect a bounded convex set K is equal to the length of its boundary.

The measure of a set of lines $G(p, \phi)$ that intersect a convex set K is defined as $m(G; G \cap K \neq \emptyset) = \int_{G \cap K \neq \emptyset} dpd\phi$.

Take a point $O \in K$ as origin.
We can take h as the support function of K with reference to O.

The support function $h_K(u)$ of a set K is the supremum of the scalar product of $x \in K$ and the argument $u \in R^d$.
$\mathrm{h}_K(u) = \sup_{x \in K} < x, u >$.

Lines that intersect a convex set

Since $\int_{G\cap K\neq\emptyset} dpd\phi = \int_0^{2\pi}\int_0^{h(\phi)} dp\, d\phi = \int_0^{2\pi} h\, d\phi$,
$m(G; G\cap K \neq \emptyset) = \int_{G\cap K\neq\emptyset} dpd\phi = \int_0^{2\pi} h\, d\phi = L$, where L is the length of the perimeter of K.

The length of a closed convex curve that has support function h of class C^2 is given by

$L = \int_0^{2\pi} h\, d\phi$.
The proof is in Santal's textbook, Integral Geometry and Geometric Probability.

Geometric probability

Suppose a line g intersects K and that K_1 is a convex set contained in the bounded convex set K.

Then, the probability that the random line intersects K_1 is L_1/L, where L_1 and L are the perimeters of K_1 and K respectively.

Theorem of Fary

Definition

If a closed plane curve of length L with absolute total curvature c_a can be enclosed by a circle of radius r, then L $\leq rc_a$.

c_a can be defined by $\int_C |d\tau|$.
τ : the angle of the tangent to C, a plane closed oriented curve of class C^2, with x axis. Then, $c_a = \int_0^\pi \nu(\tau)\, |d\tau|$.
$\nu(\tau)$: the number of unoriented tangents to C that are parallel to the direction τ. If a line G is parallel to the direction τ and meets C in n points, $P_i, i = 1, 2, 3, ..., n$, $n(\tau) \leq \nu(\tau)$.Thus
$2L = \int_{G\cap C\neq\emptyset} n\, dG \leq \int_{G\cap C\neq\emptyset} \nu\, dG = \int_{G\cap C\neq\emptyset} \nu\, dpd\tau \leq$
$2r \int_0^\pi \nu(\tau)\, |d\tau| = 2rc_a$.

References

[1] . Molchanov. Springer-Beitrag, Part 1.1.14 and 1.1.2
[2] . Solomon. Geometric probability 2^{nd} Edition. Society for Industrial Mathematics. 1987.
[3] . Santaló. Integral Geometry and Geometric Probability. Cambridge mathematical library. 2004.