Random Fields

Ying Zhang

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ulm university universität uulm

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Random elements

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\Omega \neq \emptyset$, and $(S, \mathcal{B})$ be a measurable space constructed upon an abstract set $S \neq \emptyset$
A random element $\xi : \Omega \to S$ is an $\mathcal{A}|\mathcal{B}$ -measurable mapping of $(\Omega, \mathcal{A})$ into $(S, \mathcal{B})$, i.e

$$\xi^{-1}(B) := \{\omega \in \Omega : \xi(\omega) \in B\} \in \mathcal{A}$$

for all $B \in \mathcal{B}$. We write $\xi \in \mathcal{A}|\mathcal{B}$.
Example: coin throw

Random elements

- The distribution of a random element $\xi : \Omega \to S$ is a probability measure P_ξ defined on the measurable space $(S, \mathcal{B})$ by

$$P_\xi\{B\} = P\{\xi^{-1}(B)\}, B \in \mathcal{B}.$$

- Lemma: Any probability measure μ on $(S, \mathcal{B})$ can be considered as a distribution of some random element ξ.

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Random function

- Let T be an abstract index space and $(S, \mathcal{B})$ a measurable space. A family $\xi = \{\xi(t), t \in T\}$ of random elements $\xi(t) : \Omega \to S$ defined on a probability space $(\Omega, \mathcal{A}, P)$ is called random function.
- $\xi = \xi(\omega, t)$ is a mapping of $\Omega \times T$ onto S, which is $\mathcal{A}|\mathcal{B}$-measurable for each $t \in T$.

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Random function

For $n \in N$ and $t_1, \ldots, t_n \in T$ we call the distribution of random vector $(\xi(t_1), \ldots, \xi(t_n))^\top$ a finite-dimensional distribution of the random function $\xi = \{\xi(t), t \in T\}$. We write

$$P_{t_1,\ldots,t_n}\{B_{t_1}, \ldots, B_{t_n}\} = P\{\xi(t_1) \in B_{t_1}, \ldots, \xi(t_n) \in B_{t_n}\}$$

where $B_{t_k} \in \mathcal{B}_{t_k}, k = 1, \ldots, n$

Random function

For all $n \geq 2$, $t_1, \ldots, t_n \in T$,$B_{t_k} \in E_{t_k}$, $k = 1, \ldots, n$ and all arbitrary permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, then it has the following properties:

- Symmetry:

$$P_{t_1,\ldots,t_n}\{B_{t_1} \times \ldots \times B_{t_n}\} = P_{t_{i_1},\ldots,t_{i_n}}\{B_{t_{i_1}}, \ldots, B_{t_{i_n}}\}$$

- Consistency:

$$P_{t_1,\ldots,t_n}\{B_{t_1} \times \ldots \times B_{t_{n-1}} \times E_{t_n}\} = P_{t_1,\ldots,t_{n-1}}\{B_{t_1}, \ldots, B_{t_{n-1}}\}$$

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Theorem:(A.N.Kolmogorov,1933)

Let$(E_t, \mathcal{E})_{t\in T}$ be a family of Borel spaces. For any $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$,$i \neq j$, let measures $P_{t_1,\dots,t_n}$ be given on spaces $(E_{t_1,\dots,t_n}, \mathcal{E}_{t_1,\dots,t_n})$ such that they satisfy the conditions of symmetry and consistency. Then there exist a probability space $(\Omega, \mathcal{A}, P)$ and a random function $\xi = \{\xi(t), t \in T\}$ defined on it such that its finite-dimensional distributions coincide with measures $P_{t_1,\dots,t_n}$.

White noise

A random function $\xi = \{\xi(t), t \in T\}$ defined on $(\Omega, \mathcal{A}, P)$ is called white noise, if $\xi = \{\xi(t), t \in T\}$ are independent and identically distributed.

- salt-and-pepper noise ($\xi(t) \sim Ber(p), t \in T$) for binary images.
- Gaussian white noise ($\xi(t) \sim N(0, \sigma^2), \sigma^2 > 0$) for greyscale images.

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White noise

Salt and pepper noise is a form of noise typically seen on images. It represents itself as randomly occurring white and black pixels.

Figure: The original image and the image with salt and pepper noise.

White noise

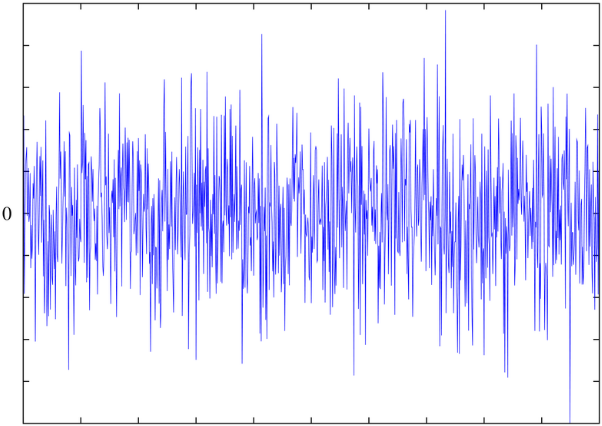

Figure: An example realization of a Gaussian white noise process.

Gaussian random function

The random function $\xi = \{\xi(t), t \in T\}$ is called Gaussian if all its finite-dimensional distributions $P_{t_1,\ldots,t_n}$ are Gaussian, which means the distribution of random vector $\xi_{t_1,\ldots,t_n} = (\xi(t_1), \ldots, \xi(t_n))^\top$ is an n-dimensional normal distribution with expectation

$$\mu_{t_1,\ldots,t_n} = (\mu(t_1), \ldots, \mu(t_n))^\top$$

and covariance matrix

$$\Sigma_{t_1,\ldots,t_n} = (cov(\xi(t_i), \xi(t_j)))_{i,j=1}^n :$$

i.e.

$$\xi_{t_1,\ldots,t_n} \sim N(\mu_{t_1,\ldots,t_n}, \Sigma_{t_1,\ldots,t_n})$$

Gaussian random function

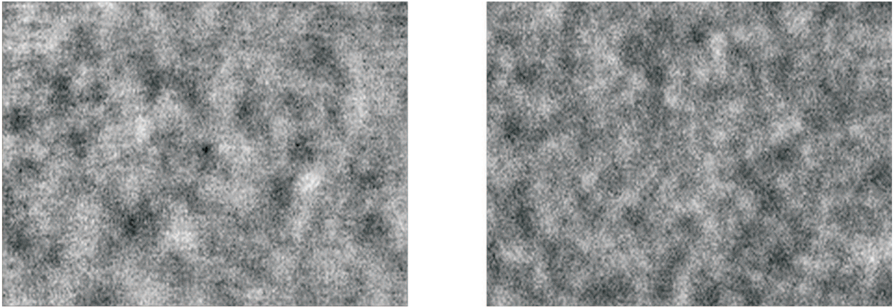

Figure: Paper surface (left) and simulated Gaussian random field (right) based on the estimated data.

Lognormal fields

The random function $\xi = \{\xi(t), t \in T\}$ is called lognormal if $\xi(t) = e^{\eta(t)}$, where $\eta = \{\eta(t), t \in T\}$ is a Gaussian random field.

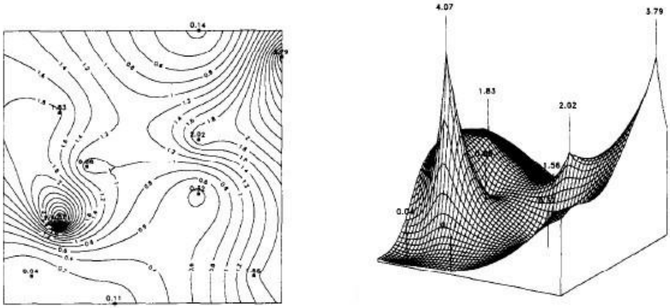

Figure: Contour maps and surface plots of a lognormal random field

χ^2-fields

The random function $\xi = \{\xi(t), t \in R^d\}$ is called χ^2-field if $\xi(t) = ||\eta(t)||_2^2, t \in R^d$, where $\eta = \{\eta(t), t \in R^d\}$ is an n-dimensional vector-valued random field such that $\eta(t) \sim N(0, I)$. I denotes the identity matrix. It is clear that $\xi(t)$ is χ_n^2-distributed for all $t \in R^d$.

Cosine fields

Let η be a random variable, ζ a random vector, $dim\zeta = dimT$ where η and ζ are independent.
Consider a random field $\xi = \{\xi(t), t \in R^d\}, d \geq 1$ defined by
$\xi(t) = \sqrt{2}\cos(2\pi\eta + \langle t, \eta\rangle)$
For instance, $\eta(t) \sim U[0,1]$, each realization of ξ is a cosine wave surface.

Cosine fields

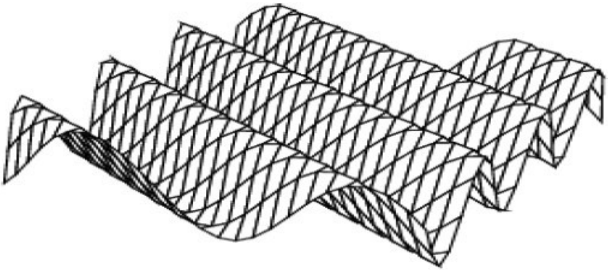

Figure: A realization of cosine field

Shot noise random fields

Let $\Pi_\lambda = x_i, i \in \mathbb{N}$ be a homogeneous Poisson point process with intensity $\lambda > 0$. Let $g : R^d \to R$ be a deterministic function, for which $\int_{R^d} g(x)dx < \infty$ and $\int_{R^d} g(x)^2 dx < \infty$ hold. If $\xi = \xi(t), t \in R^d$ by $\xi(t) = \Sigma_{x \in \Pi_\lambda} g(t - x), t \in R^d$, then ξ is called a shot-noise field and g is called response function.

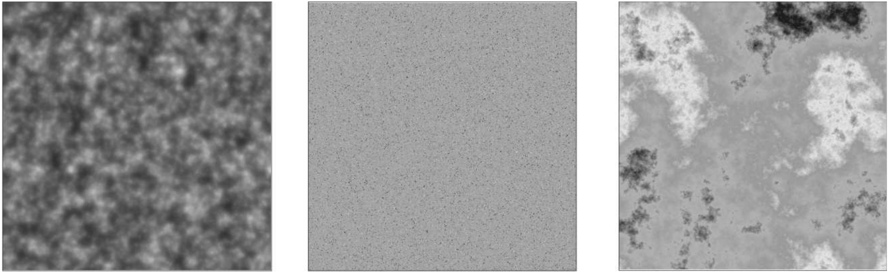

Figure: Shot-noise random field (left), Gaussian white noise (middle), Gaussian random field (right).

Construction of shot noise random fields

The response functions can be constructed as follows: take $g(x) = K(||x||_2/a)$, where $||x||_2$ is the Euclidean norm and K is called kernel which is the probability density function with compact support $K = x \in R^d : K(x) > 0$.
Formally, a shot-noise field can be written as a stochastic integral $\xi(t) = \int_{R^d} g(t-x)\Pi_\lambda(dx)$ if $\Pi_\lambda(\cdot)$ is interpreted as a random Poisson counting measure.

Moments and covariance

The mixed moment $\mu^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n)$ of ξ of orders $j_1,\ldots,j_n \in \mathbb{N}$ at index values $t_1,\ldots,t_n \in T$ is defined by:

$$\mu^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n) = E\{\xi^{j_1}(t_1)\cdot\ldots\cdot\xi^{j_1}(t_n)\}$$

For special cases:

- $\mu(t) = \mu^{(1)}(t) = E\{\xi(t)\}$
- $\mu^{(1,1)}(s,t) = E\{\xi(s)\xi(t)\}$
- $C(s,t) = Cov(\xi(s),\xi(t)) = \mu^{(1,1)}(s,t) - \mu^{(1)}(s)\mu^{(1)}(t)$

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Stationarity

- stationarity in strict sense : The random function $\xi = \{\xi(t), t \in T\}$ is called (strictly) stationary if for any $n \in \mathbb{N}, \tau, t_1, \ldots, t_n \in T$ it holds $P_{t_1+\tau,\ldots,t_n+\tau} = P_{t_1,\ldots,t_n}$, i.e. all finite-dimensional distribution of ξ are invariant with respect to shifts in T.
- stationarity in the wide sense: $\xi = \{\xi(t), t \in T\}$ is a random function with $E|\xi(t)|^2 < \infty, t \in T$. ξ is called stationarity in wide sense if $\mu(t) \equiv \mu$ and $C(s,t) = C(s+\tau, t+\tau)(:= C(s-t)), \tau, s, t \in T$.
- Both definitions of stationarity do not imply each other.

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- Both definitions of stationarity do not imply each other.

Isotropy

The random function $\xi = \{\xi(t), t \in R^d\}$ is said to be isotropic

- in the strict sense, if for any $n \in \mathbb{N}, t_1, \dots, t_n \in R^d, A \in SO(d)$:

$$(\xi(At_1), \dots, \xi(At_n))^\top \stackrel{d}{=} (\xi(t_1), \dots, \xi(t_n))^\top.$$

- in the wide sense, if for any $s, t \in R^d, A \in SO(d)$:

$$\mu(At) = \mu(t), C(As, At) = C(s, t).$$

- The random field ξ is motion invariant if it is stationary and isotropic.

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Thanks for your attention!