Junior-Prof. Dr. Z. Kabluchko
Christian Hirsch

SS 2011
June 14, 2011

Markov chains

Problem set 10

Due date: June 21, 2011

Exercise 1 [6 points]

Consider two boxes. At the beginning the first box contains k white balls and the second box contains k black balls. Every unit of time one draws one ball from the first box and independently one ball from the second box at random, all balls are equiprobable. Then, one places the ball drawn from the first box into the second box and the ball drawn from the second box into the first box. The procedure is repeated indefinitely. Let X_n be the number of white balls in the first box at time n. Find $\lim_{n\to\infty} \mathbb{P}[X_n = i]$ for $i = 0, \ldots, k$.

Exercise 2 [6 points]

A light-bulb lives Y_1 units of time. At the moment it burns out, it is immediately replaced by a new light-bulb which lives Y_2 units of time, etc. Suppose that $Y_1, Y_2, \ldots$ are independent identically distributed random variables with values in $\{1, 2, \ldots\}$ and $\mathbb{E}(Y_1) < \infty$. Let $p_n = \mathbb{P}[Y_1 = n]$ and suppose that the set of integers $\{n \in \mathbb{N} : p_n > 0\}$ has greatest common divisor 1. Let $X_n = \min\{k \in \mathbb{N}_0 : \exists m : Y_1 + \ldots + Y_m = n + k\}$ be the remaining life time of the bulb present at time n. Explain why X_n is a Markov chain and write down its transition matrix. Compute $\lim_{n\to\infty} \mathbb{P}[X_n = i]$ for $i \in \mathbb{N}_0$. What happens if $\mathbb{E}(Y_1) = \infty$?

Exercise 3 [6 points]

Consider a Markov chain X_n on the state space $E = \{1, 2, 3\}$ with initial distribution $X_0 = 1$ and transition matrix P given by

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Compute P^2, P^4 and $\lim_{n\to\infty} \mathbb{P}(X_n = i)$ for $i = 1, 2, 3$.