Exercise 1 [6 points]
Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ be two independent exponentially distributed random variables with parameters $\lambda > 0$ and $\mu > 0$. Show that for every $t > 0$,
\[
\mu P[X < t < X + Y] = \lambda P[Y < t < X + Y].
\]
Not compulsory: Can you provide a probabilistic explanation of this identity?

Exercise 2 [6 points]
Let $X_t, t \geq 0$, be a Markov chain in continuous time on state space $E = \{1, 2\}$ with generator matrix
\[
Q = \begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}, \quad \alpha > 0, \beta > 0.
\]
(a) Compute $Q^n$ for $n \in \mathbb{N}_0$ and use the formula $e^{tQ} = \sum_{n=0}^{\infty} t^n Q^n / n!$ to compute the transition probabilities $p_{ij}(t)$, $i, j \in \{1, 2\}$.
(b) Compute $\lim_{t \to +\infty} p_{ij}(t)$ for $i, j \in \{1, 2\}$.
(c) Compute $P[X_t = 2 | X_0 = 1, X_2 = 1]$.

Exercise 3 [6 points]
Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter 1. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics of $X_1, \ldots, X_n$. That is, $X_{(1)}$ is the smallest, $X_{(2)}$ the second smallest, \ldots, $X_{(n)}$ the largest of $\{X_1, \ldots, X_n\}$. Set also $X_{(0)} = 0$.
(a) Show that the random variables $Y_i := X_{(i+1)} - X_{(i)}$, $i = 0, \ldots, n-1$ are independent and $Y_i \sim \text{Exp}(n-i)$. (Hint: Consider a death process starting with $n$ individuals).
(b) Show that $E[\max(X_1, \ldots, X_n)] = \sum_{k=1}^{n} \frac{1}{k}$.
(c) Let $Z_1, Z_2, \ldots$ be independent exponential random variables with parameter 1. Show that the following convergence in distribution holds:
\[
\frac{1}{k} \sum_{k=1}^{n} Z_k - \log n \xrightarrow{n \to \infty} G,
\]
where $G$ is a random variable with Gumbel distribution: $P[G \leq t] = e^{-e^{-t}}$, $t \in \mathbb{R}$.
(d) (Not compulsory) Compute $E G$. 