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SS 2011
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Markov chains

Problem set 11

Due date: June 28, 2011

Exercise 1 [6 points]

Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ be two independent exponentially distributed random variables with parameters $\lambda > 0$ and $\mu > 0$. Show that for every $t > 0$,

$$\mu \mathbb{P}[X < t < X + Y] = \lambda \mathbb{P}[Y < t < X + Y].$$

Not compulsory: Can you provide a probabilistic explanation of this identity?

Exercise 2 [6 points]

Let $X_t, t \geq 0$, be a Markov chain in continuous time on state space $E = \{1, 2\}$ with generator matrix

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}, \qquad \alpha > 0, \beta > 0.$$

(a) Compute Q^n for $n \in \mathbb{N}_0$ and use the formula $e^{tQ} = \sum_{n=0}^{\infty} t^n Q^n / n!$ to compute the transition probabilities $p_{ij}(t)$, $i, j \in \{1, 2\}$.

(b) Compute $\lim_{t \to +\infty} p_{ij}(t)$ for $i, j \in \{1, 2\}$.

(c) Compute $\mathbb{P}[X_t = 2 | X_0 = 1, X_{2t} = 1]$.

Exercise 3 [6 points]

Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter 1. Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistics of $X_1, \ldots, X_n$. That is, $X_{(1)}$ is the smallest, $X_{(2)}$ the second smallest,..., $X_{(n)}$ the largest of $\{X_1, \ldots, X_n\}$. Set also $X_{(0)} = 0$.

(a) Show that the random variables $Y_i := X_{(i+1)} - X_{(i)}$, $i = 0, \ldots, n-1$ are independent and $Y_i \sim \text{Exp}(n - i)$. (Hint: Consider a death process starting with n individuals).

(b) Show that $\mathbb{E}[\max(X_1, \ldots, X_n)] = \sum_{k=1}^{n} \frac{1}{k}$.

(c) Let $Z_1, Z_2, \ldots$ be independent exponential random variables with parameter 1. Show that the following convergence in distribution holds:

$$\sum_{k=1}^{n} \frac{Z_k}{k} - \log n \xrightarrow[n \to \infty]{d} G,$$

where G is a random variable with Gumbel distribution: $\mathbb{P}[G \leq t] = e^{-e^{-t}}$, $t \in \mathbb{R}$.

(d) (Not compulsory) Compute $\mathbb{E}G$.