Junior-Prof. Dr. Z. Kabluchko
Christian Hirsch

SS 2011
June 28, 2011

Markov chains

Problem set 12

Due date: July 5, 2011

Please solve at most 3 problems. You can obtain 6 points for every problem. The problems starting with problem 4 are more difficult than problems $1, 2, 3$. If you would like to solve more than 3 problems please show the solutions directly to Zakhar or Christian.

Exercise 1

A fair die (with 6 sides) is thrown repeatedly. Let S_n be the sum of numbers shown up in the first n throws. Let $T = \min\{n \in \mathbb{N} : S_n \text{ is divisible by } 5\}$ be the first time the sum becomes divisible by 5. Compute $\mathbb{E}T$, the expectation of T.

Exercise 2

Consider a random walker on the set $\{1, \dots, k\}$ (where $k \geq 3$) who goes from a state $i \in \{2, \dots, k-1\}$ to $i+1$ or $i-1$ both with probability $1/2$. Assume also that states 1 and k are repelling which means that if the random walker is at 1 (respectively, k), then he goes to 2 (respectively, $k-1$) with probability 1. Let X_n be the position of the walker at time n and suppose that $X_0 = 1$. Compute $\lim_{n\to\infty} \mathbb{P}[X_{2n} = i]$ for $i \in \{1, 3, \dots, 2\ell - 1\}$, where $\ell = \lfloor (k+1)/2 \rfloor$.

Exercise 3 (Markov queue with infinitely many servers)

Customers arrive at a service point with intensity λ. The customers are served by infinitely many servers (so that each customer finds a free server immediately after arrival). Each server can serve only one customer simultaneously. The service time of a customer is exponentially distributed with parameter μ. Let X_t be the number of customers which are served at time t. Compute $\lim_{t\to\infty} \mathbb{P}[X_t = n]$ for $n \in \mathbb{N}_0$.

Exercise 4 (Birth process with immigration)

Consider an island which is not inhabited at time 0. New individuals arrive to the island with rate $\nu > 0$. Every individual which is on the island gives birth to new individuals with rate $\lambda > 0$. All individuals behave independently of each other. Let X_t be the number of individuals on the island at time t. Compute $m(t) := \mathbb{E}[X_t]$.

Exercise 5 [Discrete Liouville theorem]

Consider a Markov chain in discrete time with transition matrix P. A function $f : E \to \mathbb{R}$

is called harmonic if for every $i \in E$,

$$f(i) = \sum_{j \in E} p_{ij} f(j).$$

(a) Show that if the chain is recurrent and irreducible, then every positive harmonic function is constant.

(b) Show that if the chain is recurrent and irreducible, then every bounded harmonic function is constant.

Exercise 6 [Discrete Liouville theorem]

A function $f : \mathbb{Z}^d \to \mathbb{R}$ is called harmonic if its value at every $x \in \mathbb{Z}^d$ is equal to the arithmetic mean of its values on the $2d$ neighboring sites of x.

(a) Is it true that every positive harmonic function is constant?

(b) Is it true that every bounded harmonic function is constant?

Exercise 7 [Branching process with two types]

A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 1, becomes mature. Each mature cell, after an exponential time of parameter 2 divides into two immature cells. At time 0 there is one immature cell. Let $n(t)$ (respectively, $m(t)$) be the expected number of immature (respectively, mature) cells at time t. Compute $n(t)$ and $m(t)$.

Exercise 8 [Voter process]

Consider an $n \times n$ array of squares. Every square which is not on the boundary of the array is adjacent to 4 other squares. We make the following convention, called periodic boundary condition. If a square S is (say) on the left-hand boundary of the array, it is adjacent to the square on the right-hand boundary in the same row as S. A similar assumption is made about squares on the upper and lower boundaries. With this convention, every square in the array is adjacent to exactly 4 other squares.

At every square a voter is located who may vote for one of two parties, called red and black. At time 0, there were a red and b black voters, $a + b = n^2$. At any time, a voter is chosen at random, all voters being equiprobable. This voter then chooses one of its 4 neighbors at random (all neighbors are equiprobable) and assumes the opinion (color) of that neighbor. The colors of all other voters remain unchanged. Then, the procedure is repeated. Compute the probability that there will be a moment of time at which all voters are black.