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SS 2011
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Markov chains

Problem set 13

Please do not hand in this sheet

All students who wish to participate in the final written exam should register at LSF-website (Hochschulportal) that they successfully completed the exercise classes (*Vorleistung*). The registration deadline is July 8, 2011. A link can be found on this lecture's website
With respect to the overall difficulty the final exam corresponds approximately to Problems $1-4$ and $5a$ of this exercise sheet.

Exercise 1

Three out of every 4 trucks on the road are followed by a car, while only 1 out of every 5 cars is followed by a truck. What fraction of vehicles on the road are trucks? (You may assume that every vehicle is either car or truck and use an appropriate Markov chain model).

Exercise 2

Consider a tournament between two teams. The first team is from city A, the second one from city B. The tournament consists of a series of independent matches that alternately take place in city A and city B (the first match takes place in city A).

In its hometown, the team from city A wins a match against team B with probability $1/2$. However, in city B, the team from city A wins a match only with probability $1/3$. The tournament ends as soon as one team has won 2 *consecutive* matches (and this team is declared the winner of the tournament).

(a) Compute the probability that team A wins the tournament.

(b) Compute the expected number of matches of the tournament.

Exercise 3

Consider a 8×8 chess board. A king may move to any (horizontally, vertically or diagonally) neighboring square. At time 0 the king starts at the lower left square of the board and then performs a random walk. At each time the king is at a certain square, all the admissible moves are equiprobable.

(a) Compute the expected time of first return to the lower left corner of the board.

(b) Compute the expected number of visits to the upper right square of the board before the first return to the lower left corner.

Exercise 4

Consider a (non-simple) random walk on the state space $\{0, 1, 2, \ldots\}$. The transition probabilities are given by

$$p_{k,k+1} = \frac{2}{3} \text{ for } k \geq 0; \qquad p_{k,k-1} = \frac{1}{3} \text{ for } k > 0; \qquad p_{0,0} = \frac{1}{3}.$$

Prove that this Markov chain is irreducible and transient.

Exercise 5

Customers arrive at a service point with intensity 1. Every customer wishes to be served first by server A and then by server B (in this order). Every server can serve only one customer simultaneously. The service time of a customer served by server A (resp. B) is exponentially distributed with parameter 2 (resp. 3). When a customer arrives at the service point, she goes to server A. If server A is already occupied, she leaves the service point without being served. The customers who have been served by server A proceed to server B. If server B is already occupied, the customer leaves the service point without being served completely. Customers who have been served by server B leave the service point.

(a) Let X_t denote the set of servers occupied at time t. Compute $\lim_{t\to\infty} \mathbb{P}(X_t = S)$ for all $S \subset \{A, B\}$.

(b) Let N_t be the number of customers who arrived at the service point during the interval $[0, t]$ without receiving the complete service. Compute $\lim_{t\to\infty} \mathbb{E}N_t/t$.