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SS 2011
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Markov chains

Problem set 2

Due date: April 26, 2011

Exercise 1 [6 points]

Three girls A, B and C are playing table tennis. In each game, two of the girls are playing against each other and the third girl does not play. In game $n+1$, the winner of game n plays against the girl which did not participate in game n. The probability that girl x beats girl y in any game is $s_x/(s_x+s_y)$, where $x, y \in \{A, B, C\}$, $x \neq y$, and $s_A, s_B, s_C > 0$ represent the strengths of the girls. Consider a Markov chain whose state at time n is the pair of girls playing at that time.

(a) Construct the transition matrix of this Markov chain.

(b) Determine the probability that the two girls who play each other in the first game will play each other again in the fourth game. Show that this probability does not depend on which two girls play in the first game. (Hint: Start by computing this probability under the assumption that the first players are A and B. Then show that your result is symmetric in A, B, C).

Exercise 2 [6 points]

A fair die is rolled repeatedly. Let N_n be the largest number shown in the rolls $1, 2, \ldots, n$ (define also $N_0 = 0$). Show that $N_0, N_1, \ldots$ is a Markov chain and compute its transition matrix.

Exercise 3 [6 points]

Suppose that weather can be either sunny or rainy. If the weather is sunny on one day, then the next day will be sunny with probability $1-p$ and rainy with probability p. If the weather is rainy on one day, then the next day will be rainy with probability $1-p'$ and sunny with probability p'. Suppose that $0 < p, p' < 1$.

(a) Show that the n-step transition matrix of the corresponding Markov chain is equal to

$$P^n = \frac{1}{p+p'}\begin{pmatrix} p' & p \\ p' & p \end{pmatrix} + \frac{(1-p-p')^n}{p+p'}\begin{pmatrix} p & -p \\ -p' & p' \end{pmatrix}.$$

(b) Show that $\lim_{n\to\infty} P^n = \frac{1}{p+p'}\begin{pmatrix} p' & p \\ p' & p \end{pmatrix}$.

For the assessment of your homework solutions a registration at SLC is required!