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SS 2011
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Markov chains

Problem set 4

Due date: May 10, 2011

Exercise 1 [6 points]

Three robots A, B, C fight a battle. Robot A has probability $1/2$ of destroying the robot at which it fires, robot B has probability $1/3$ of destroying the robot at which it fires, and robot C has probability $1/6$ of destroying the robot at which it fires. The robots fire simultaneously and each robot fires at the strongest opponent not yet destroyed (A is stronger than B, B is stronger than C). The fight continues as long as at least two robots remain undestroyed. Let X_n be the set of robots which remain undestroyed at time n. For example, $X_0 = \{A, B, C\}$.

(a) Write down the transition matrix of the Markov chain X_n. Show that the states $\{A\}, \{B\}, \{C\}$ and $\emptyset$ are absorbing.

(b) Compute the probability q_A (respectively, q_B, q_C) that robot A (respectively, B, C) wins the battle.

Hint. To verify your result, C is *most* likely to win and B is *least* likely to win. The probability that at the end all robots are destroyed is $q_\emptyset = 17/182$.

Exercise 2 [6 points]

A fair die is rolled repeatedly. Denote by X_n the number of different numbers observed in the rolls $1, \ldots, n$. For example, if the rolls are $3, 5, 3, 1, 4, 6, 6, 2, \ldots$, then $X_1 = 1$, $X_2 = 2$, $X_3 = 2$, $X_4 = 3$, $X_5 = 4$, $X_6 = 5$, $X_7 = 5$ and $X_k = 6$ for $k \geq 8$.

(a) Write down the transition matrix of the Markov chain X_n and show that 6 is an absorbing state.

(b) Let $N = \min\{n \in \mathbb{N} : X_n = 6\}$ be the first time at which all 6 numbers have been observed. Compute $\mathbb{E}N$, the expectation of N.

Exercise 3 [6 points]

A fair coin whose sides are denoted by H (heads) and T (tails) is dropped repeatedly. Let N be the time at which the pattern HTH is observed for the first time. For example, if the outcomes are $HTTHTTHTH\ldots$, then $N = 9$. Compute $\mathbb{E}N$, the expectation of N.

Hint. Consider a Markov chain whose states are initial segments of the pattern HTH, that is $E = \{\emptyset, H, HT, HTH\}$. Interpret N as an absorption time at state HTH.