Markov chains
Problem set 8
Due date: June 7, 2011

Exercise 1 [6 points]

Assume that a man's profession can be classified as professional, skilled laborer, or unskilled laborer. Assume that, of the sons of professional men, 80% are professional, 10% are skilled laborers, and 10% are unskilled laborers. In the case of sons of skilled laborers, 60% are skilled laborers, 20% are professional, and 20% are unskilled. Finally, in the case of unskilled laborers, 50% of the sons are unskilled laborers, and 25% each are in the other two categories. Assume for simplicity that every man has exactly one son. Find the limit, as $n \to \infty$, of the probability that the descendant in the n-th generation of an unskilled laborer is a professional man.

Exercise 2 [6 points]

A fair die is thrown repeatedly. Let S_n be the sum of the first n throws. Compute

$$\lim_{n \to \infty} \mathbb{P}[S_n \text{ is a multiple of 7}].$$

Hint. Consider the remainder of S_n modulo 7 as a Markov chain.

Exercise 3 [6 points]

A number of customers are waiting for service in a waiting line. Each unit of time either one customer departs or one new customer arrives. Given that there are $i \in \{0, 1, \ldots\}$ customers in the waiting line, the probability that a customer departs is $\frac{\lambda i}{\lambda i + \mu}$ and the probability that one new customer arrives is $\frac{\mu}{\lambda i + \mu}$, where $\lambda, \mu > 0$. Let X_n be the number of customers waiting in the line at time n and let $X_0 = 0$. Compute $\lim_{n \to \infty} \mathbb{P}[X_{2n} = 2i]$, $i \in \mathbb{N}_0$.

Exercise 4 [Not compulsory, 6 points]

Assume that a student going to a certain four-year school has, each year, a probability 0.1 of flunking out, and probability 0.2 of having to repeat the year, and a probability 0.7 of moving on to the next year. In the fourth year, moving on means graduating.

(a) Form a transition matrix for this Markov chain taking as states $F, 1, 2, 3, 4, G$ where F stands for flunking out and G for graduating, and the other states represent the year of study. Find the communicating classes of this Markov chain. Which of communicating classes are recurrent?

(b) Find the probability that the student will graduate and the probability that he will leave the school without graduating.

(c) How long should this student expect to be in the school?