Markov chains
Problem set 9
Due date: June 14, 2011

Exercise 1 [6 points]
A queen on a $4 \times 4$-chessboard makes each permissible move with equal probability. If it starts in a corner of the chessboard, how long on average will it take to return to the corner?

Exercise 2 [6 points]
Suppose that there are $N$ molecules in a box which is divided into two equal halves by a partition. A small hole is made in the partition. Suppose that at any moment of time one of the molecules is chosen at random (all $N$ molecules are equiprobable) and moves through the hole in the partition to the other half of the box. Let $X_n$ be the number of particles in the left half of the box at time $n$. Compute the unique invariant probability distribution of the Markov chain $X_n$ and show that this chain is reversible.

Exercise 3 [6 points]
Consider a Markov chain on the state space $\{1, \ldots, d\}$ such that the states $m + 1, \ldots, d$ are absorbing (that is, $p_{jj} = 1$ for $j = m + 1, \ldots, d$) and for every state $i \in \{1, \ldots, m\}$ there is an absorbing state $j \in \{m + 1, \ldots, d\}$ and a number $k$ such that $p_{ij}^{(k)} > 0$. Here $0 < m < d$. The transition matrix of this chain can be written in the form

$$ P = \begin{pmatrix} Q & R \\ 0 & 1 \end{pmatrix}, $$

where $Q$ is a $m \times m$-matrix, $R$ is an $m \times (d - m)$ and $1$ is the $(d - m) \times (d - m)$ identity matrix.

(a) Show that $\lim_{n \to \infty} Q^n = 0$.

(b) Consider the matrix $N = (1 - Q)^{-1}$ and show that the $ij$-entry of $N$ is the expected number of times the Markov chain visits $j$ given that it starts at $i$, where $i, j \in \{1, \ldots, m\}$.

Exercise 4 [Not compulsory]
Consider an irreducible, positive recurrent Markov chain (which, however, need not be aperiodic). Denote by $\{\pi_i : i \in E\}$ the unique invariant probability distribution. Show that for every $i, j \in E$,

$$ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)} = \pi_j. $$