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SS 2011
June 7, 2011

Markov chains

Problem set 9

Due date: June 14, 2011

Exercise 1 [6 points]

A queen on a 4×4-chessboard makes each permissible move with equal probability. If it starts in a corner of the chessboard, how long on average will it take to return to the corner?

Exercise 2 [6 points]

Suppose that there are N molecules in a box which is divided into two equal halves by a partition. A small hole is made in the partition. Suppose that at any moment of time one of the molecules is chosen at random (all N molecules are equiprobable) and moves through the hole in the partition to the other half of the box. Let X_n be the number of particles in the left half of the box at time n. Compute the unique invariant probability distribution of the Markov chain X_n and show that this chain is reversible.

Exercise 3 [6 points]

Consider a Markov chain on the state space $\{1, \dots, d\}$ such that the states $m+1, \dots, d$ are absorbing (that is, $p_{jj} = 1$ for $j = m+1, \dots, d$) and for every state $i \in \{1, \dots, m\}$ there is an absorbing state $j \in \{m+1, \dots, d\}$ and a number k such that $p_{ij}^{(k)} > 0$. Here $0 < m < d$. The transition matrix of this chain can be written in the form

$$P = \begin{pmatrix} Q & R \\ 0 & \mathbf{1} \end{pmatrix},$$

where Q is a $m \times m$-matrix, R is an $m \times (d-m)$ and $\mathbf{1}$ is the $(d-m) \times (d-m)$ identity matrix.

(a) Show that $\lim_{n\to\infty} Q^n = 0$.

(b) Consider the matrix $N = (1-Q)^{-1}$ and show that the ij-entry of N is the expected number of times the Markov chain visits j given that it starts at i, where $i, j \in \{1, \dots, m\}$.

Exercise 4 [Not compulsory]

Consider an irreducible, positive recurrent Markov chain (which, however, need not be aperiodic). Denote by $\{\pi_i : i \in E\}$ the unique invariant probability distribution. Show that for every $i, j \in E$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p_{ij}^{(k)} = \pi_j.$$