Write $G_n = \sum_{k=1}^{n} \frac{Z_k}{k} - \log(n)$. By part 3c, and [1, Lemma 3.11] it suffices to prove that the family $\{G_n\}_{\geq 1}$ is uniformly integrable. First observe that

$$\gamma_n := \mathbb{E}(G_n) = \sum_{k=1}^{n} \frac{1}{k} - \log(n) \to \gamma,$$

where γ is the Euler-Mascheroni constant. Now we can compute

$$\begin{aligned}
\mathbb{E}(|G_n|\, 1_{|G_n|>r}) &\leq \mathbb{E}(|G_n - \gamma_n|\, 1_{|G_n|>r}) + \gamma_n \mathbb{P}(|G_n| > r) \\
&\leq \mathbb{E}(|G_n - \gamma_n|\, 1_{|G_n-\gamma_n|>r/2}) + \mathbb{E}(|G_n - \gamma_n|\, 1_{|\gamma_n|>r/2}) + \\
&+ \gamma_n \mathbb{P}(|G_n - \gamma_n| > r/2) + \gamma_n \mathbb{P}(\gamma_n > r/2) \\
&\leq \frac{Var(G_n)}{r/2} + \mathbb{E}(|G_n - \gamma_n|\, 1_{|\gamma_n|>r/2}) + \frac{\gamma_n Var(G_n)}{(r/2)^2} + \gamma_n \mathbb{P}(\gamma_n > r/2)
\end{aligned}$$

Thus for all $r > 2\gamma$ we have

$$\limsup_n \mathbb{E}(|G_n|\, 1_{|G_n|>r}) \leq (\sum_{k=1}^{\infty} \frac{1}{k^2})(\frac{1}{r/2} + \frac{\gamma}{(r/2)^2}) \xrightarrow{r\to\infty} 0$$

References

[1] O. Kallenberg. *Foundations of modern probability.* Springer Verlag, 2002.