Problem 3

Let us denote by $(X_n)_{n\geq 0}$ the Markov chain describing the position of the random walker. Then $(X_n)_{n\geq 0}$ is clearly irreducible and positively recurrent (as it is irreducible and the state space is finite). Thus the expected time of first return is given by $E(T) = \frac{1}{\alpha_{(1,1)}}$, where $\alpha^T = (\alpha_{(x,y)})_{1\leq x,y\leq 8}$ is the unique invariant probability measure for the chain $(X_n)_{n\geq 0}$. Furthermore, since $(X_n)_{n\geq 0}$ describes a simple random walk on a finite graph, we have

$$\alpha_{(1,1)} = d_{(1,1)} \left(\sum_{\substack{(x,y)\\1\leq x,y\leq 8}} d_{(x,y)} \right)^{-1}.$$

Where $d_{(x,y)}$ denotes the degree of the vertex at position (x,y). Now one can easily check that

$$\begin{aligned} &d_{(1,1)} = d_{(1,8)} = d_{(8,1)} = d_{(8,8)} = 3 \\ &d_{(1,x)} = d_{(8,x)} = d_{(x,1)} = d_{(x,8)} = 5 \qquad \text{for all } 2\leq x\leq 7 \\ &d_{(x,y)} = 8 \qquad \text{for all } 2\leq x,y\leq 7 \end{aligned}$$

Thus we have

$$\sum_{\substack{(x,y)\\1\leq x,y\leq 8}} d_{(x,y)} = 4\cdot 3 + 24\cdot 5 + 36\cdot 8 = 420$$

implying that $\alpha_{(1,1)} = \frac{1}{140}$ and therefore $\mathbb{E}(T) = 140$.

The expected number of visits in state $(8,8)$ is given by the quotient $\frac{\alpha_{(8,8)}}{\alpha_{(1,1)}} = 1$.

Problem 5b

First we consider an informal argument which gives us the correct solution (don't do this in the exam!). Observe that no customers are rejected in states $\emptyset$ and $\{B\}$. While the chain is in state $\{A\}$, all newly arriving customers are rejected. Since customers arrive at rate 1, the number of customers that leave without receiving the full service grows at rate 1 while in state $\{A\}$. In state $\{A,B\}$ newly arriving customers are rejected, too. Furthermore, if server A finishes serving his customer while the chain is in state $\{A,B\}$ (which happens with intensity 2), this customer will leave the service point without receiving full service. Thus, while in state $\{A,B\}$ customers will leave without receiving full service at rate 3. Now, by the results of part a, in the long run the chain spends $\frac{1}{30}$ of the time in state $\{A,B\}$ and $\frac{3}{10}$ of the time in state $\{A\}$. Thus the overall rate at which customers are rejected is given by $1\cdot\frac{3}{10} + 3\cdot\frac{1}{30} = \frac{2}{5}$.

Now to the formal solution: Consider the Markov chain on $E = 2^{\{A,B\}}\times\mathbb{N}_0$ describing the joint state of the current set of occupied servers and the number of customers that walked away without receiving full service. Then the non-zero entries of the generator Q are given as follows

$$q((\emptyset,n),(\{A\},n)) = 1 \qquad q((\emptyset,n),(\emptyset,n)) = -1$$

$$q((\{A\},n),(\{B\},n)) = 2 \qquad q((\{A\},n),(\{A\},n+1)) = 1 \qquad q((\{A\},n),(\{A\},n)) = -3$$
$$q((\{B\},n),(\emptyset,n)) = 3 \qquad q((\{B\},n),(\{A,B\},n)) = 1 \qquad q((\{B\},n),(\{B\},n)) = -4$$
$$q((\{A,B\},n),(\{A\},n)) = 3 \qquad q((\{A,B\},n),(\{B\},n+1)) = 2$$
$$q((\{A,B\},n),(\{A,B\},n+1)) = 1 \qquad q((\{A,B\},n),(\{A,B\},n)) = -6$$

Let us write $m_\emptyset(t) = \mathbb{E}(N_t)$ the expected number of customers that have left without being served completely. Analogously write $m_A(t), m_B(t), m_{A,B}(t)$ for the expected number of customers that have left by time t without being served completely, when the MC starts in state $(\{A\},0)$, $(\{B\},0)$ resp. $(\{A,B\},0)$. Using the Kolmogorov differential equation $(P(t)' = QP(t))$, we obtain

$$\begin{aligned} m_\emptyset(t)' &= \sum_{S\subset\{A,B\}}\sum_{n\geq 0} n\cdot p_{(\emptyset,0),(S,n)}(t)' \\ &= \sum_{S\subset\{A,B\}}\sum_{n\geq 0} n\cdot(-p_{(\emptyset,0),(S,n)}(t) + p_{(\{A\},0),(S,n)}) \\ &= m_A(t) - m_\emptyset(t) \end{aligned}$$

Similarly, one obtains:

$$\begin{aligned} m_A(t)' &= 2m_B(t) + (m_A(t)+1) - 3m_A(t) \\ m_B(t)' &= 3m_\emptyset(t) + m_{AB}(t) - 4m_B(t) \\ m_{AB}(t)' &= 3m_A(t) + 2(m_B(t)+1) + (m_{AB}(t)+1) - 6m_{AB}(t) \end{aligned}$$

Solving this system of ODEs unter the IV-conditions ($m_\emptyset(0) = m_A(0) = m_B(0) = m_{AB}(0)$) gives the solutions:

$$\begin{aligned} m_\emptyset(t) &= \frac{1}{3}e^{-3t} - \frac{1}{4}e^{-4t} + \frac{2}{25}e^{-5t} + \frac{2}{5}t - \frac{49}{300} \\ m_A(t) &= -\frac{2}{3}e^{-3t} + \frac{3}{4}e^{-4t} - \frac{8}{25}e^{-5t} + \frac{2}{5}t + \frac{71}{300} \\ m_B(t) &= \frac{1}{3}e^{-3t} - \frac{3}{4}e^{-4t} + \frac{12}{25}e^{-5t} + \frac{2}{5}t - \frac{19}{300} \\ m_{AB}(t) &= -\frac{2}{3}e^{-3t} + \frac{3}{4}e^{-4t} - \frac{18}{25}e^{-5t} + \frac{2}{5}t + \frac{191}{300} \end{aligned}$$

In particular $\mathbb{E}(N_t)/t = m_\emptyset(t)/t \to \frac{2}{5}$.