1 Aufgabe 4

According to the exercise text, the transition matrix is given by $P = (p_{i,j})_{i,j\in\mathbb{N}_0}$ with

$$p_{i,j} = \begin{cases} \frac{1}{i+2} & \text{, if } j \leq i+1 \\ 0 & \text{else} \end{cases}$$

Writing down the equation $\alpha^T P = \alpha^T$ explicitly yields the following system of equations

$$\sum_{j=0}^{\infty} 1/(j+2)\alpha_j = \alpha_0$$
$$\sum_{j=i-1}^{\infty} 1/(j+2)\alpha_j = \alpha_i \text{ for } i \geq 1$$

We claim that any such sequence must satisfy $\alpha_n = n! \cdot \alpha_0$. This can be proved by induction, the cases $n = 0, 1$ being clear. If this assertion is already true for m, then it follows that

$$\frac{1}{m+1}\alpha_{m-1} + \alpha_{m+1} = \alpha_m$$

from which we obtain

$$\begin{aligned}
\alpha_{m+1} &= \alpha_m - \frac{1}{m+1}\alpha_{m-1} \\
&= \frac{1}{m!}\alpha_0 - \frac{1}{m+1}\frac{1}{(m-1)!}\alpha_0 \\
&= \frac{1}{(m-1)!}(\frac{1}{m} - \frac{1}{m+1})\alpha_0 \\
&= \frac{1}{(m+1)!}\alpha_0
\end{aligned}$$

The condition $\sum_{n=0}^{\infty} \alpha_n = 1$ finally implies $\alpha_0 = e^{-1}$ and it is easy to check that $\alpha_n = (n! \cdot e)^{-1}$ is indeed an invariant probability measure.

To compute the expectation, we first note that

$$\begin{aligned}
\mathbb{E}(X_{n+1}|X_n) &= \frac{1}{X_n+2}(\sum_{k=0}^{X_n+1} k) \\
&= \frac{X_n+1}{2}
\end{aligned}$$

Using the tower property of conditional expectations one can prove by induction that for all $k \leq n$ the relation

$$\mathbb{E}(X_{n+1}|X_k) = \frac{X_k + 2^{n-k+1} - 1}{2^{n-k+1}}$$

holds. In particular

$$\mathbb{E}(X_{n+1}) = \frac{2^{n+1}-1}{2^{n+1}}$$