1 Aufgabe 2

As in the hint to the exercise, we define $X_n = S_n \mod 7$. Then $(X_n)_{n\geq 0}$ defines a MC with transition matrix

$$P = \frac{1}{6}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Again the MC is clearly irreducible and positively recurrent. Furthermore, it is aperiodic since for all $i \in \{0, \ldots, 6\}$ we have $\mathbb{P}_i(X_2 = i) \geq \frac{1}{6^2} > 0$ and also $\mathbb{P}_i(X_3 = i) \geq \frac{1}{6^3} > 0$. Thus an application of the ergodic theorem yields

$$\lim_{n\to\infty} \mathbb{P}(S_n \text{ is a multiple of } 7) = \lim_{n\to\infty} \mathbb{P}_0(X_n = 0) = \alpha_0,$$

where $\alpha = (\alpha_0, \ldots, \alpha_6)^T$ is the unique invariant probability measure. Solving the system $\alpha^T P = \alpha^T$ shows that α^T is given by $\alpha_i = \frac{1}{7}$ for all $i \in \{0, \ldots, 6\}$. Hence

$$\lim_{n\to\infty} \mathbb{P}(S_n \text{ is a multiple of } 7) = \frac{1}{7}$$

2 Aufgabe 3

Let $\rho = \frac{\mu}{\lambda}$. Then the transition matrix of the MC $(X_n)_{n\geq 0}$ is given by $P = (p_{ij})_{i,j\in\mathbb{N}_0}$ where

$$\begin{aligned} p_{0,j} &= 1_{j=1} \text{ for all} j \geq 0 \\ p_{i,j} &= 1_{j=i-1}\frac{i}{i+\rho} + 1_{j=i+1}\frac{\rho}{i+\rho} \text{ for all } i \geq 1, j \geq 0 \end{aligned}$$

This can be visualized as

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{1+\rho} & 0 & \frac{\rho}{1+\rho} & 0 & 0 & \cdots \\ 0 & \frac{2}{2+\rho} & 0 & \frac{\rho}{2+\rho} & 0 & \cdots \\ 0 & 0 & \frac{3}{3+\rho} & 0 & \frac{\rho}{3+\rho} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Unfortunately though, this MC is not aperiodic.

However, if we write $P^2 = (q_{ij})_{i,j\in\mathbb{N}_0}$, then it is easy to check that $q_{2i,2j+1} = 0$ for all $i, j \geq 0$. Thus we can define a new MC $(Y_n)_{n\geq 0}$ on the state space $E' = 2\mathbb{N}_0 = \{0, 2, 4, \ldots\}$ with $Y_0 = 0$ and transition matrix $P' = (q_{2i,2j})_{2i,2j\in 2\mathbb{N}_0}$. Since $X_0 = 0 \in E'$ we have $(Y_n)_{n\geq 0} \overset{d}{=} (X_{2n})_{n\geq 0}$. Since $q_{2i,2i} > 0$ for all $i \geq 0$ the MC $(Y_n)_{n\geq 0}$ is aperiodic and it is easy to check that it is irreducible. Furthermore, to check positive recurrence, we look for invariant probability measures.

First observe that if α is a distribution on $\mathbb{N}_0$ satisfying $\alpha^T P = \alpha^T$, then also $\alpha^T P^2 = \alpha$. Since $q_{2j+1,2i} = 0$ for all $i, j \geq 0$, we furthermore obtain $\alpha' = (\alpha_0, \alpha_2, \alpha_4, \ldots)$ also satisfies $(\alpha')^T P' = (\alpha')^T$.

Writing out the equation $\alpha^T P = \alpha^T$ explicitly yields

$$\begin{aligned} \alpha_1 \frac{1}{1+\rho} &= \alpha_0 \\ \alpha_{i-1}\frac{\rho}{i-1+\rho} + \alpha_{i+1}\frac{i+1}{i+1+\rho} &= \alpha_i \text{ for all } i \geq 1 \end{aligned}$$

First we claim that this implies $\frac{i}{i+\rho}\alpha_i = \frac{\rho}{i-1+\rho}\alpha_{i-1}$ for all $i \geq 1$. This is proven by induction. For $i = 1$ the assertion is immediate and for the induction step, we obtain

$$\begin{aligned} \frac{i+1}{i+1+\rho}\alpha_{i+1} &= \alpha_i - \frac{\rho}{i-1+\rho}\alpha_{i-1} \\ &= \alpha_i - \frac{i}{i+\rho}\alpha_i \\ &= \frac{\rho}{i+\rho}\alpha_i \end{aligned}$$

Using this representation another induction argument shows $\alpha_n = \frac{\rho+n}{n}\frac{\rho^{n-1}}{(n-1)!}\alpha_0$. Again it is easy to check that for any α satisfying these relations the equation $\alpha^T P = \alpha^T$ will hold.

As discussed above this also implies that $(\alpha')^T P' = (\alpha')^T$ holds. Using the relation $\sum_{i=0}^{\infty} \alpha_{2i} = 1$, we obtain

$$1 = \alpha_0(1 + \sum_{i=1}^{\infty}(1 + \frac{\rho}{2i})\frac{\rho^{2i-1}}{(2i-1)!})$$
$$1 = \alpha_0 e^{\rho},$$

i.e. $\alpha_{2n} = e^{-\rho}(\frac{\rho^{2n}}{(2n)!} + \frac{\rho^{2n-1}}{(2n-1)!})$. Since $(Y_n)_{n\geq 0}$ has a strictly positive invariant probability measure it is positively recurrent and this probability measure is in fact unique.

Now we have checked all the assumptions for the ergodic theorem and we obtain

$$\lim_{n\to\infty} \mathbb{P}_0(X_{2n} = 2i) = \lim_{n\to\infty} \mathbb{P}_0(Y_n = 2i) = \alpha_{2i} = e^{-\rho}(\frac{\rho^{2i}}{(2i)!} + \frac{\rho^{2i-1}}{(2i-1)!})$$

3 Aufgabe 4

3.1 4a

The transition matrix is given as follows (where G is state 5 and F is state 6)

$$P = \begin{pmatrix} 0.2 & 0.7 & 0 & 0 & 0 & 0.1 \\ 0 & 0.2 & 0.7 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.7 & 0 & 0.1 \\ 0 & 0 & 0 & 0.2 & 0.7 & 0.1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus the communicating classes are given by {F}, {G} and {1},{2},{3},{4}. Both {F} and {G} are trivially recurrent. However {1},{2},{3},{4} are not recurrent, since there is a positive probability of moving to the absorbing state and thus never returning to one of the class {1},{2},{3},{4} again.

3.2 4b

To compute the probability of eventually graduating, we need to solve the following system of equations

$$h_1^{(G)} = 0.2h_1^{(G)} + 0.7h_2^{(G)}$$
$$h_2^{(G)} = 0.2h_2^{(G)} + 0.7h_3^{(G)}$$
$$h_3^{(G)} = 0.2h_3^{(G)} + 0.7h_4^{(G)}$$
$$h_4^{(G)} = 0.2h_4^{(G)} + 0.7$$

Doing this gives $h_1^{(G)} = \frac{2401}{4096}$ and $h_1^{(F)} = 1 - h_1^{(G)} = \frac{1695}{4096}$.

3.3 4c

To compute the expected value, we need to solve the following set of equations

$$k_1 = 0.2k_1 + 0.7k_2 + 1$$
$$k_2 = 0.2k_2 + 0.7k_3 + 1$$
$$k_3 = 0.2k_3 + 0.7k_4 + 1$$
$$k_4 = 0.2k_4 + 1$$

Doing this gives $k_1 = \frac{8475}{2048}$.