Risk Theory

Exercise Sheet 2

Due to: 4th May 2012

Hint: Exercise sheets should be handed in by a team of exactly two students!

Exercise 1 (6 points) Distribution function of the Erlang distribution
Let $X \sim \text{Gamma}(k, \lambda)$ be a Gamma–distributed random variable, where $\lambda > 0$ and $k > 0$ is integer. Show that the distribution function of $X$ is given by

$$F_X(t) = 1 - \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \quad t \geq 0.$$ 

Remark. The Gamma($k, \lambda$) distribution with $k \in \mathbb{N}$ is the distribution of the arrival time of the $k$-th claim in the Poisson point process with intensity $\lambda$. It is also called the Erlang distribution.

Exercise 2 (6 points) The mode of the Gamma distribution
For a random variable $X$ with a continuous density function $f_X$, the mode is defined as the value $t_{\text{mode}} \in \mathbb{R}$ with the property $f_X(t_{\text{mode}}) = \sup_{t \in \mathbb{R}} f_X(t)$, provided such value exists and is unique. The mode can be interpreted as the “most probable” value of $X$. Let $X \sim \text{Gamma}(\alpha, \lambda)$, where $\alpha > 1$ and $\lambda > 0$. Compute the mode of $X$. What happens if $\alpha \in (0, 1]$?

Exercise 3 (6 points) Convolution property of binomial and Poisson distributions
(a) Let $X_1 \sim \text{Bin}(n_1, p)$ and $X_2 \sim \text{Bin}(n_2, p)$ be two independent random variables, where $n_1, n_2 \in \mathbb{N}$ and $p \in [0, 1]$. Show that $X_1 + X_2 \sim \text{Bin}(n_1 + n_2, p)$.

(b) Let $Y_1 \sim \text{Poi} (\lambda_1)$ and $Y_2 \sim \text{Poi} (\lambda_2)$ be two independent random variables, where $\lambda_1, \lambda_2 > 0$. Show that $Y_1 + Y_2 \sim \text{Poi} (\lambda_1 + \lambda_2)$.

Exercise 4 (7 points) Bernoulli claim arrival process
Let $\xi_1, \xi_2, \ldots$ be independent and identically Bernoulli distributed random variables with parameter $p \in (0, 1]$. Consider the Bernoulli claim arrival process $T_0 = 0$, $T_1 = \min \{ k \in \mathbb{N} : \xi_k = 1 \}$, $T_2 = \min \{ k > T_1 : \xi_k = 1 \}, \ldots$, $T_n = \min \{ k > T_{n-1} : \xi_k = 1 \}$. Prove that the increments $W_1 := T_1 - T_0, W_2 := T_2 - T_1, \ldots, W_n := T_n - T_{n-1}$ are independent and identically distributed random variables having a geometric distribution with parameter $p$. 
