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# Risk Theory

## Exercise Sheet 4

Due to: 25th May 2012

**Exercise 1** (6 points)

Let $X$ be a random variable with finite Laplace transform. The cumulants $\kappa_n(X)$ of $X$ are defined by

$$\kappa_n(X) = \frac{d^n}{dt^n} \log(\mathbb{E}(e^{tX}))|_{t=0}$$

(a) Show that $\kappa_1(X) = \mathbb{E}[X]$, $\kappa_2(X) = \mathrm{Var}\,(X)$, $\kappa_3(X) = \mathbb{E}[(X - \mathbb{E}[X])^3]$.

(b) Show that if $X, Y$ are independent random variables, then $\kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y)$.

(c) Calculate the cumulants of the Poisson distribution with parameter $\lambda$ and the normal distribution with parameters $\mu, \sigma^2$.

Hint: You may interchange the derivative and the expectation sign without justifying it.

**Exercise 2** (6 points)

Suppose that the number of claims is a mixed Poisson variable with mean 10000 and standard deviation 1000. Find the standard deviation of the mixing variable.

**Exercise 3** (6 points)

The Laplace transform (or moment-generating function) $L_X$ of a random variable $X$ is given by $L_X(t) = \mathbb{E}[e^{tX}]$ for all $t \in \mathbb{R}$ for which the expected value exists. The generating function $g_N$ of a random variable $N : \Omega \to \mathbb{N}_0$ is given by $g_N(s) = \mathbb{E}[s^N]$, for $|s| \leq 1$.

(a) Let $X \sim \mathrm{Exp}(\lambda)$, with $\lambda > 0$. Compute $L_X$.

(b) Let $N \sim \widetilde{\mathrm{NB}}(\gamma, p)$ for some $\gamma > 0$, $p \in (0,1)$. Show by using generating functions that $\mathbb{E}[N] = \gamma\frac{1-p}{p}$ and $\mathrm{Var}\,(N) = \gamma\frac{1-p}{p^2}$.

**Exercise 4** (6 points)

Let $X = \sum_{i=1}^{N} U_i$ be a compound Poisson random variable with $\mathbb{E}[U_i^2] < \infty$ and $N \sim \mathrm{Poi}(\lambda)$. Prove the following central limit theorem: for $\lambda \to \infty$

$$\frac{X - \mathbb{E}[X]}{\sqrt{\mathrm{Var}\,(X)}} \xrightarrow{d} Y \sim N(0,1)$$

Hint: Compute the characteristic function of the left-hand side and take the limit $\lambda \to \infty$.

**Exercise 5** (6 points)

Consider a collective model with the number of claims $N \sim \text{Poi}(\lambda)$ and the claim size $X_1$ having a logarithmic distribution with parameter $p \in (0,1)$, that is

$$\mathbb{P}(X_1 = n) = -\frac{1}{\log(1-p)} \frac{p^n}{n}, \; n \in \mathbb{N}$$

Show that the total claim amount $S = X_1 + ... + X_N$ satisfies $S \sim \widetilde{\text{NB}}\left(-\frac{\lambda}{\log(1-p)}, 1-p\right)$.

**Exercise 6** (6 points)

(a) Let $X$ be a $\text{G}(\alpha, \lambda)$-distributed random variable. Prove that

$$\frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}\,(X)}} \xrightarrow{d} Y \sim N(0,1)$$

as $\alpha \to \infty$ and $\lambda$ stays constant.
Hint: Compute the characteristic function of the left-hand side.

(b) Let $X_n$ be a random variable with $\text{G}(cn, 1)$-distribution, where $c > 0$ is constant. Show that $\frac{X_n}{n}$ converges, as $n \to \infty$, in distribution towards a random variable which takes value $c$ with probability one.
Hint: Compute the limit of $\varphi_{X_n/n}(t)$, as $n \to \infty$.

**Exercise 7** (6 points)

A positive random variable $X$ has *inverse Gaussian distribution* with parameter $\mu, \lambda > 0$, denoted by $X \sim IG(\mu, \lambda)$, if the density function is given by

$$f_X(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \; x > 0.$$

Show that the distribution function of $X$ is

$$F_X(x) = \Phi\left(\frac{\sqrt{\lambda}}{\sqrt{x}}\left(\frac{x}{\mu} - 1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\frac{\sqrt{\lambda}}{\sqrt{x}}\left(\frac{x}{\mu} + 1\right)\right), \; x > 0.$$

Hint: It is not necessary to integrate.