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Stochastic networks II

Problem set 1

Due date: April 24, 2012

Let $r > 0$ and let $\{s_n\}_{n\geq 1} \subset \mathbb{R}^2$ be a locally finite, countable set. Furthermore denote by $G(\{s_n\}_{n\geq 1}, r)$ the geometric graph on the vertex set $\{s_n\}_{n\geq 1}$ whose edge set is given by $\{\{s_n, s_m\} : |s_n - s_m| \leq r\}$

Exercise 1

Let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1.

(a) Let $r > 0$. Show $\lim_{a\to\infty} \mathbb{P}\left(\exists X_n \in X \cap [-a/2, a/2]^2 : X \cap B_r(X_n) = \{X_n\}\right) = 1$, where $B_r(X_n) = \{y \in \mathbb{R}^2 : |X_n - y| \leq r\}$.

(b) Prove

$$\lim_{a\to\infty} \mathbb{P}\left(\bigcap_{\substack{z\in\mathbb{Z}^2 \\ \frac{r_a}{4}(z+[-1/2,1/2]^2)\cap[-a/2,a/2]^2\neq\emptyset}} \left\{\frac{r_a}{4}(z + [-1/2, 1/2]^2) \cap X \neq \emptyset\right\} \right) = 1,$$

where $r_a = 8\log(a)$.

Exercise 2

Let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1.

(a) Let $r > 0$. Prove $\lim_{a\to\infty} \mathbb{P}(G(X \cap [-a/2, a/2]^2, r)$ is connected$) = 0$.

(b) Prove $\lim_{a\to\infty} \mathbb{P}(G(X \cap [-a/2, a/2]^2, r_a)$ is connected$) = 1$, where $r_a = 8\log(a)$.

Exercise 3

Let $r > 0$ and let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1.

(a) Show that the conditional distribution of X given the event $X([-\delta/2, \delta/2]^2) = 1$ is given by the distribution of $\{S\} \cup Y$, where $S \sim U([-\delta/2, \delta/2]^2)$ and Y is the restriction to $\mathbb{R}^2 \setminus [-\delta/2, \delta/2]^2$ of an homogeneous Poisson point process of intensity 1 which is independent of S.

(b) Prove $\lim_{\delta\to 0} \mathbb{P}\left(\exists X_n \in X : B_r(X_n) \cap [-\delta/2, \delta/2]^2 \neq \emptyset \text{ and } [-\delta/2, \delta/2]^2 \not\subset B_r(X_n)\right) = 0$

(c) For $\delta > 0$ write A_δ for the event that there is an infinite component of $G(X, r)$ which has non-empty intersection with $[-\delta/2, \delta/2]^2$. Prove $\mathbb{P}(A_\delta \mid X([-\delta/2, \delta/2]^2) = 1) \xrightarrow{\delta\to 0} \theta(r)$, where $\theta(r)$ denotes the probability that the origin is contained in an infinite connected component of $G(X \cup \{o\}, r)$.