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SS 2012
April 24, 2012

Stochastic networks II

Problem set 2

Due date: May 8, 2012

Exercise 1

Let $X = \{S_n\}_{n\geq 1} \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1 and for $k \geq 1$ let $Q_1^{(k)}, \ldots, Q_{k^2}^{(k)}$ be a subdivision of $[0,1]^2$ into congruent squares of side length $1/k$. For $r > 0$ write $G(X,r) = (X, E_r)$ for the geometric graph with vertex set X and edge set $E_r = \{\{v, v'\} \subset X : 0 < |v - v'| \leq r\}$. Furthermore write $X_k = \sum_{i=1}^{k^2} X_i^{(k)}$, where for $1 \leq i \leq k^2$ we denote by $X_i^{(k)}$ the indicator function of the event that precisely one point of $X \cap Q_i^{(k)}$ is contained in an infinite connected component of $G(X,r)$.

(a) ~~Prove $X_k \leq X_{k+1}$ for each $k \geq 1$.~~

(b) Prove $\lim_{k\to\infty} X_k = N_r([0,1]^2)$, where $N_r([0,1]^2)$ denotes the number of points of $X \cap [0,1]^2$ that are contained in an infinite connected component of $G(X,r)$.

Exercise 2

Let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. Write $r_c = \inf\{r : \theta(r) > 0\}$ and for every $\alpha \in (0,1)$ write $r_\alpha = \inf\{r : \theta(r) > \alpha\}$.

(a) Let $r > 0$ be arbitrary. Prove $\theta(r) = \lim_{n\to\infty} \mathbb{P}(C_o^{(r)} \not\subset [-n/2, n/2]^2)$, where $C_o^{(r)}$ denotes the connected component of $G(X \cup \{o\}, r)$ containing o.

(b) Show that the function $\theta : (0,\infty) \to [0,1]$, $r \mapsto \theta(r)$ is continuous from the right.

(c) Conclude $r_c < r_\alpha$ for every $\alpha > 0$.

(d) Prove $r_\alpha < \infty$ for all $\alpha \in (0,1)$.

Hint. You may use $\theta(r_c) = 0$ without proof. See also Stochastic Networks I, problem set 5, exercise 3.

Exercise 3

Let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. For $r > 0$ denote by $X_r^{(\infty)} \subset X$ the points of X that are contained in an infinite connected component of $G(X,r)$ and for $A \subset \mathbb{R}^2$ bounded Borel write $N_r(A) = \#(X_r^{(\infty)} \cap A)$. For $k \geq 1$, $\delta > 0$ write $B_{\delta,k} = [-\sqrt{\delta k}/2, \sqrt{\delta k}/2]^2$.

(a) Prove $\int_{B_{\delta,k}} N_r(x + [0,1]^2)dx = \sum_{X_n \in X_r^{(\infty)}} \nu_2((X_n + [-1,0]^2) \cap B_{\delta,k})$, where ν_2 denotes two-dimensional Lebesgue measure.

(b) Prove

$$\lim_{k\to\infty} \frac{1}{\delta k} \sum_{X_n \in X_r^{(\infty)}} 1_{(X_n+[-1,0]^2)\cap B_{\delta,k}\neq\emptyset \text{ and } (X_n+[-1,0]^2)\not\subset B_{\delta,k}} = 0 \text{ a.s.}$$

(c) Conclude

$$\lim_{k\to\infty} \frac{N_r(B_{\delta,k})}{\delta k} = \lim_{k\to\infty} \frac{\int_{B_{\delta,k}} N_r(x+[0,1]^2)dx}{\delta k} = \mathbb{E}(N_r([0,1]^2)) \text{ a.s.}$$

Hint. Use the Borel-Cantelli Lemma for part (*b*) and ergodic theory for the second equation in part (*c*).

Exercise 4 *

Let X, $X_r^{(\infty)}$, N_r and $B_{\delta,k}$ be as in problem 3. The goal of the present problem is to show

$$\lim_{k\to\infty} \frac{1}{(\delta k)^2} \mathrm{Var} N_r(B_{\delta,k}) = 0.$$

(a) For $A \subset \mathbb{R}^2$ Borel write $N_r'(A) = X(A) - N_r(A)$. Reduce the problem to proving

$$\lim_{k\to\infty} \frac{1}{(\delta k)^2} \mathrm{Var} N_r'(B_{\delta,k}) = 0.$$

(b) Subdivide $B_{\delta,k}$ into $M_k = \lceil 2\sqrt{\delta k}/r \rceil^2$ squares $Q_1, \dots, Q_{M_k}$ of side length $\in (r/4, r/2)$. Prove

$$\mathrm{Var}(N_r'(B_{\delta,k})) = \sum_{1\le i,j\le M_k} \mathrm{Cov}(N_r'(Q_i), N_r'(Q_j))$$

(c) Prove that

$$\Big| \sum_{1\le i\le M_k} \sum_{\substack{1\le j\le M_k \\ d(Q_i,Q_j)\le k^{1/3}}} \mathrm{Cov}(N_r'(Q_i), N_r'(Q_j)) \Big| \le k^{7/4}$$

holds for all sufficiently large $k \ge 1$, where we write $d(Q_i, Q_j) = \inf\{d(x,y) : x \in Q_i, y \in Q_j\}$.

(d) Prove that for all $1 \le i,j \le M_k$ we have

$$\mathrm{Cov}(N_r'(Q_i), N_r'(Q_j)) = \mathrm{Cov}(N_r'(Q_i)1_{|C_i|<\infty}, N_r'(Q_j)1_{|C_j|<\infty}),$$

where C_i denotes the union of all connected components of $G(X,r)$ that contain a point from Q_i.

(e) Prove that

$$\left|\mathrm{Cov}(N_r'(Q_i)1_{C_i\not\subset Q_i\oplus[-k^{1/4},k^{1/4}]}1_{|C_i|<\infty}, N_r'(Q_j))\right| \le \exp(-k^{1/5})$$

for all sufficiently large k (use the hint and Cauchy-Schwarz inequality)

(f) Prove the existence of $k_0 \geq 1$ such that

$$\mathrm{Cov}(N'_r(Q_i)1_{C_i \subset Q_i \oplus [-k^{1/4}, k^{1/4}]}, N'_r(Q_j)1_{C_j \subset Q_j \oplus [-k^{1/4}, k^{1/4}]}) = 0,$$

holds for all $k \geq k_0$ and all $1 \leq i, j \leq M_k$ with $d(Q_i, Q_j) > k^{1/3}$.

(g) Conclude

$$\lim_{k \to \infty} \frac{1}{(\delta k)^2} \mathrm{Var} N'_r(B_{\delta,k}) = 0.$$

Hint. You may use the following result without proof. Let $r > r_c$ and let $A \subset \mathbb{R}^2$ be a bounded Borel set. Then there exists $c > 0$ such that for all sufficiently large $k \geq 1$ we have

$$\mathbb{P}(\mathrm{diam}(C_A) \geq k, |C_A| < \infty) \leq \exp(-ck),$$

where C_A denotes the union of the connected components of $G(X, r)$ containing a point from A and where $\mathrm{diam}(C_A) = \sup\{d(x, y) : x, y \in C_A\}$.