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Stochastic networks II

Problem set 3

Due date: May 15, 2012

Exercise 1

Let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. For $r > 0$ write $G(X \cup \{o\}, r) = (X \cup \{o\}, E_r)$ for the geometric graph with vertex set $X \cup \{o\}$ and edge set $E_r = \{\{v, v'\} \subset X \cup \{o\} : 0 < |v - v'| \leq r\}$. Write $r_c = \inf\{r : \theta(r) > 0\}$.

(a) Show that for $r < r_c$ with probability 1 all connected components of the graph $G(X \cup \{o\}, r)$ are finite.

(b) Show that for $r > r_c$ with probability 1 there exists an infinite connected component of the graph $G(X \cup \{o\}, r)$.

Hint. Use ergodic theory.

Exercise 2

Let X, $G(X \cup \{o\}, r)$ and r_c be as in exercise 1.

(a) show that for all $r, r' \geq 0$ with $r_c < r < r'$ we have

$$\mathbb{P}\left(\left|C_o^{(r')}\right| = \infty, \left|C_o^{(r)}\right| < \infty\right) > 0.$$

(b) conclude that the function $\theta : [0, \infty) \to [0, 1]$ is strictly increasing on the interval (r_c, ∞).

Exercise 3

Let X be as in exercise 1 and let $(A_k)_{k\geq 1}$ be an arbitrary sequence of bounded Borel sets of $\mathbb{R}^2$ with $\nu_2(A_k) \to \infty$ as $k \to \infty$. Show the stochastic convergence $X(A_k)/\nu_2(A_k) \to 1$.

Exercise 4

For $r > 0$ let us consider the random site process $\{Y_z\}_{z\in\mathbb{Z}^2}$ defined by $Y_z(\omega) = 1_{X\cap(r/4\cdot(z+[0,1]^2))=\emptyset}(\omega)$. We say that z is activated if $Y_z = 1$. Furthermore for $z \in \mathbb{Z}^2$ write C_z for the connected component of activated sites of $\mathbb{Z}^2$ containing z (here we mean connectedness using the 8-neighborhood).

(a) Show that there exist $c, r_0 > 0$ such that $\mathbb{P}(|C_z| > n) \leq 2\exp(-cn)$ holds for all $n \geq 1$ and all $r \geq r_0$.

(b) Let $k, \ell \geq 1$ be arbitrary. Show that for $r \geq r_0$ the probability for the existence of an activated vertical crossing of the discrete rectangle $R = ([0, kn] \times [0, \ell n]) \cap \mathbb{Z}^2$ tends to 0 as $n \to \infty$. Here we say R has an activated vertical crossing if there exists an activated path in R whose starting point is contained in $([0, kn] \times \{0\}) \cap \mathbb{Z}^2$ and whose end point is contained in $([0, kn] \times \{\ell n\}) \cap \mathbb{Z}^2$.

(c) Conclude that for all $k, \ell \geq 2$ and all $r \geq r_0$ the probability that $G(X, r)$ defines a horizontal crossing of the rectangle $[0, (k-1)nr/4] \times [0, (\ell+1)nr/4]$ tends to 1 as $n \to \infty$.